Perturbative + Semiclassical Methods

- Perturbation theory in Schwinger-Keldysh formalism
- Secular terms and interpretation
- Boltzmann approach
- Boltzmann-Vlasov
- Infrared issues, classical field theory

Schwinger-Keldysh Perturbation theory

Two sets of fields, currents: ϕ_+, ϕ_-

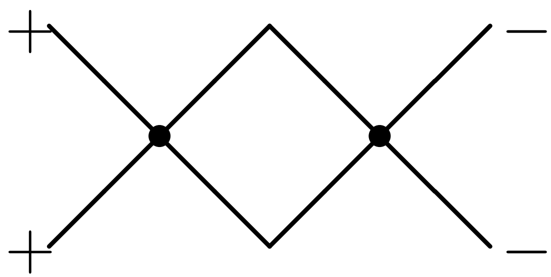
Two sets of interaction vertices: $\mathcal{L}_+$ and $\mathcal{L}_-$

Two-by-two propagator: $G_{\pm\pm}$:

$$\begin{aligned}
G_{++}(Q) \equiv G_{\rm T}(Q) &=_{\rm equil} \frac{-i}{Q^2 - i\epsilon} + 2\pi\delta(Q^2) n(|q^0|) {\rm sgn}(q^0) \\
G_{+-}(Q) \equiv G^{>}(Q) &=_{\rm equil} 2\pi (n(q^0) \pm 1) \delta(Q^2) \\
G_{-+}(Q) \equiv G^{<}(Q) &= G^*_{+-}(-Q), \\
G_{--}(Q) \equiv G_{\overline{\rm T}} &= G^*_{\rm T}(-Q) \,.
\end{aligned}$$

In every diagram, sum over each internal vertex being all-$+$ or all-$-$ [with $-$ sgn]. $2^{n_{\rm vert}}$ times more work!

Example:



Sum: $(+-)$, $(-+)$, $(++)$, $(--)$

Interpretation: $++$, $--$ correct vertices, $+-$ represents scattering, $-+$ represents???

Reduces to vacuum theory if $n_b \to 0$:

- $G^{>}$ represent final state particles (cut)
- G_{++}, G_{--} are propagators in $\mathcal{M}$, $\mathcal{M}^*$.

Interpretation

- G_{+-} counts on-shell excitations
- $G_{\rm R} \equiv G_{++} - G_{+-} = G_{-+} - G_{--}$ off-shell propagation.

In equilibrium, $G^{>}(Q) = (n(q^0) \pm 1)\ {\rm Disc} G_{\rm R}$

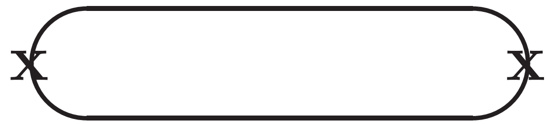
Best to use avg $r = [(+) + (-)]/2$, diff $a = [(+) - (-)]$ variables: $G_{aa} = 0$ and

$$\begin{aligned} G_{rr} &= (n \pm 1/2) 2\pi \delta(Q^2) \\ G_{\rm R} = G_{ra} &= \frac{-i}{Q^2} \quad \text{with } \ q^0 \to q^0 + i\epsilon \end{aligned}$$

Vertices have odd # of a's. Vac. interp...

Problem: secular terms.

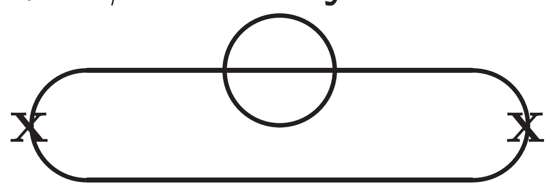
Consider 2-pt function of T_{xy} in real time. Leading diagram

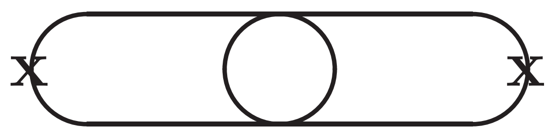


$\int d^3x$ gives time-independent contribution!

(If on-shell particles are there, they remain forever)

One-loop corrections, $\lambda\phi^4$ theory:





give resp. negative, positive $\lambda^2 t$ contributions

Secular terms

As such, disastrous. Pert. series is

$$\langle T_{xy} T_{xy} \rangle \sim c_0 + c_1 \lambda^2 t T + c_2 \lambda^4 t^2 T^2 + \ldots$$

does not converge for time $t > 1/\lambda^2 T$. Why?

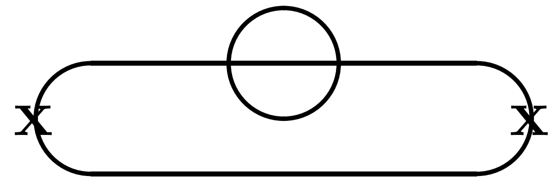
One diagram simple: self-energy represents particle loss by scattering. We know how to resum it: do so

$$\langle T_{xy} T_{xy} \rangle \sim c_0 e^{-\lambda^2 t} + c_1 \lambda^2 t e^{-\lambda^2 t} + c_2 \lambda^4 t^2 e^{-\lambda^2 t} + \ldots$$

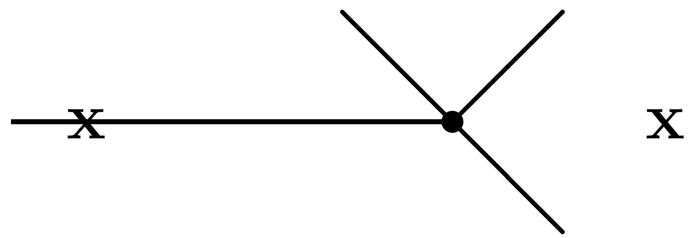
Now finite, but still fails to converge.

Problem is particle propagation. Need to resum it.

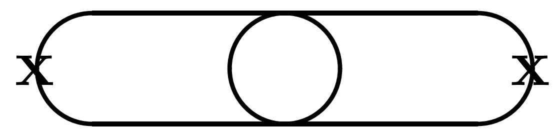
first diagram



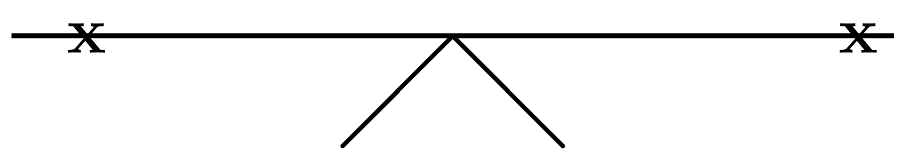
Destructive interference between propagation (lower line) and scattering away from propagation:



Other diagram



disturbance due to one of scattered particles:



Kinetic (Boltzmann) approach

Particles described by $G^{>}$. In noneq. setting,

$$G^{>}(x,y) = \langle \phi(y)\phi(x) \rangle$$

Write equation of motion:

$$\partial_x^2 - \partial_y^2 G_{+-}(x,y) = \sum_i \int_z \Big(G_{+i}(x,z)\Sigma_{i-}(z,y) - \Sigma_{+i}(x,z)G_{i-}(z,y) \Big)$$

Change coord to average and difference
$x, y \rightarrow X + r/2, X - r/2$
and Fourier transform WRT relative coord r:

$$\partial_x^2 - \partial_y^2 = 2\partial_X \partial_r = 2ip^\mu \partial_\mu^X$$

Boltzmann cont.

Now make approximations: On-shell propagation:

$$G^{>}(p,x) = \frac{\pi}{\omega_p}\Big[(n(p,x)\pm 1)\delta(p^0-\omega_p) + n(-p,x)\delta(p^0+\omega_p)\Big]$$

Slow variation in space: Assume $G^{>}$'s in self-energy given at same x as $G^{>}$

Expand self-energy to some order–use some set of collisions

$$\begin{aligned} 2p^\mu\partial_\mu n(p,x) \;\; &= \;\; -\int_{kp'k'}(2\pi)^4\delta^4(P{+}K{-}P'{-}K')|\mathcal{M}^2| \times \\ &\qquad \Big(n(p)n(k)[1{\pm}n(p')][1{\pm}n(k')] \\ &\qquad\quad -n(p')n(k')[1{\pm}n(p)][1{\pm}n(k)]\Big) \end{aligned}$$

Discussion

You could have guessed most of this.

- $$2p^\mu \partial_\mu = (2E)\Big(\partial_t + \vec{v} \cdot \vec{\nabla}_x\Big)$$

  "convective deriv": propagation v times spatial inhomog. causes time change.

- $1/2E \times$ collision term is σ times scatterer flux.

- $n(p)n(k)$ initial occupancies, $[1 \pm n(p')][1 \pm n(k')]$ are (Bose stimulation/Pauli blocking) factors

- $[1 \pm n(p)] \ldots$ term is "gain" term, particles scattering *into* momentum state p

Background $F_{\mu\nu}$?

Need to remember

- $n(p)$ really $n_{a\bar{a}}(p) = \langle \phi^\dagger_{\bar{a}} \phi_a \rangle$
- ∂_μ really D_μ

Pick up a commutator term:

$$2p^\mu \partial_\mu n \longrightarrow 2p^\mu D_\mu n + g p^\mu \Big\{ F_{\mu\nu} \, , \, \partial_p^\nu n \Big\}$$

(Known for ages, eg Vlasov. Careful derivation: Blaizot Iancu hep-ph/9903389)

But what makes me think there should be classical field anyway?

Schwinger-Keldysh: another interpretation

The propagator $G_{\mathrm{R}}(P)$ represents free propagation.
Valid for classical fields, quantum excitations, anything.

The correlation function $G_{rr}(P) = (n(p) + 1/2)2\pi\delta(P^2)$ describes vacuum $(1/2)$ plus particle $n(p)$ fluctuations.

Each vertex has odd number of a's since $\mathcal{L}_+ - \mathcal{L}_-$

Hence one extra G_{rr} per loop order.

Vaccum: loops count powers in quantum fluctuations:
Manifest by associating $\hbar$ with G_{rr}.

Finite T: just adding particle fluctuations on top of vac.

Schwinger-Keldysh and Classical fields

Consider distribution function in IR region $E \ll T$:

$$n_b(p) = \frac{1}{e^{E/T} - 1} \quad \text{really} \quad \frac{1}{e^{\hbar\omega_p}T - 1}$$

Expand in small $\hbar\omega_p/T$:

$$\frac{1}{2} + n_b(p) \sim \frac{T}{\hbar\omega_p} - \frac{\hbar\omega_p}{12T} + \ldots$$

Leading term is $1/\hbar$! large occupancy is classical fields.

Corrections down by two powers of $\hbar$, small for $\omega < 2T$.

One $G_{rr} \sim n_b + 1/2$ per loop: no $\hbar$ per loop!

Classical field approximation

With $n_b + 1/2 \rightarrow T/\hbar\omega_p$ approx,
Pert thy and classical field pert thy are identical Aarts hep-ph/9707342

Treat IR region using classical field thy!

We also need to: one factor of n_b per loop:

$$c_1 + c_2 \alpha_{\rm s} n_b(\omega) + c_3 \alpha_{\rm s}^2 n_b^2(\omega)$$

fails to converge for $\hbar\omega \sim \alpha_{\rm s} T$.

IR region is nonperturbative!

Solving classical field theory

We need fully nonperturbative technique: lattice!

$$D_\mu F^{\mu\nu} = 0$$

is nonlinear as $D = \partial - iA$ and $F_{\mu\nu} = -i\Big[\, D_\mu \,,\, D_\nu \,\Big]$.

This equation of motion arose by extremizing action

$$\frac{\partial S}{\partial A_\nu(x)} = 0\,, \qquad S = \int d^4x \frac{1}{2g^2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$$

Need lattice implementation of A_μ and of S.

Classical lattice gauge theory 1

Numerical methods require finite # of DOF.

Make space finite and discretize it,

$$x_i \longrightarrow a n_i \,, \quad n_i \in [0, 1, \dots N)$$

identifying N with 0 (periodic boundaries).

Space spacing a. Time spacing a_t (in a moment)

Should I then write $A_\mu(x) = A_\mu(a n_i + a_t n_t)$?

NO!

Classical lattice gauge theory 2

Observation (Wilson '74): essential to keep gauge invariance.

Gauge invariance: indexed fields

$$\psi_a(x) = \begin{bmatrix} \psi_r(x) \\ \psi_g(x) \\ \psi_b(x) \end{bmatrix} \quad \text{invariant under } \psi_a(x) \to U_{a\bar{b}}(x)\psi_b(x)$$

with $U_{a\bar{b}}(x) = R_{a\bar{b}}(g(x))$ rep matrix of group element $g(x) \in \mathcal{G}$ (say, $\mathcal{G} = \mathrm{SU}(3)$)

Essential: INDEPENDENT rotations at each point in space.

Classical lattice gauge theory 3

Must be able to compare fields at different points:

Need comparator, called Wilson line:

$W_{C:a\bar{b}}(x,y)\psi_b(y)$ acts like it's at x, in sense

$$W_{C:a\bar{b}}(x,y)\psi_b(y) \to U_{a\bar{c}}(x)W_{C:c\bar{b}}(x,y)\psi_b(y)$$

This requires W transform as

$$W_{C:a\bar{b}}(x,y) \to U_{a\bar{c}}(x)W_{C:c\bar{d}}(x,y)U^{-1}_{d\bar{b}}(y)$$

Still not unique: must specify path $C : y \to x$.

Classical lattice gauge theory 4

Assuming $W_{C:a\bar{c}}$ generated from something local:
must be of form

$$W_{C:a\bar{b}}(x,y) = \left(\mathrm{Pexp}\int_{C:x}^{y} -iA_\mu^A T^A dl^\mu\right)_{a\bar{b}}$$

with $T^A_{a\bar{b}}$ gen. matrices of representation R

Infinitesimal form:

$$W_{a\bar{b}}(x, x+\epsilon^\mu) = \delta_{a\bar{b}} - i\epsilon^\mu A_\mu^A T^A_{a\bar{b}}$$

Include when taking derivatives:

$$D_\mu \psi_a \equiv \frac{W(x, x+\epsilon\hat{\mu})\psi(x+\epsilon\hat{\mu}) - \psi(x)}{\epsilon} = (\partial_\mu \delta_{a\bar{b}} - iA_\mu^A T^A_{a\bar{b}})\psi_b$$