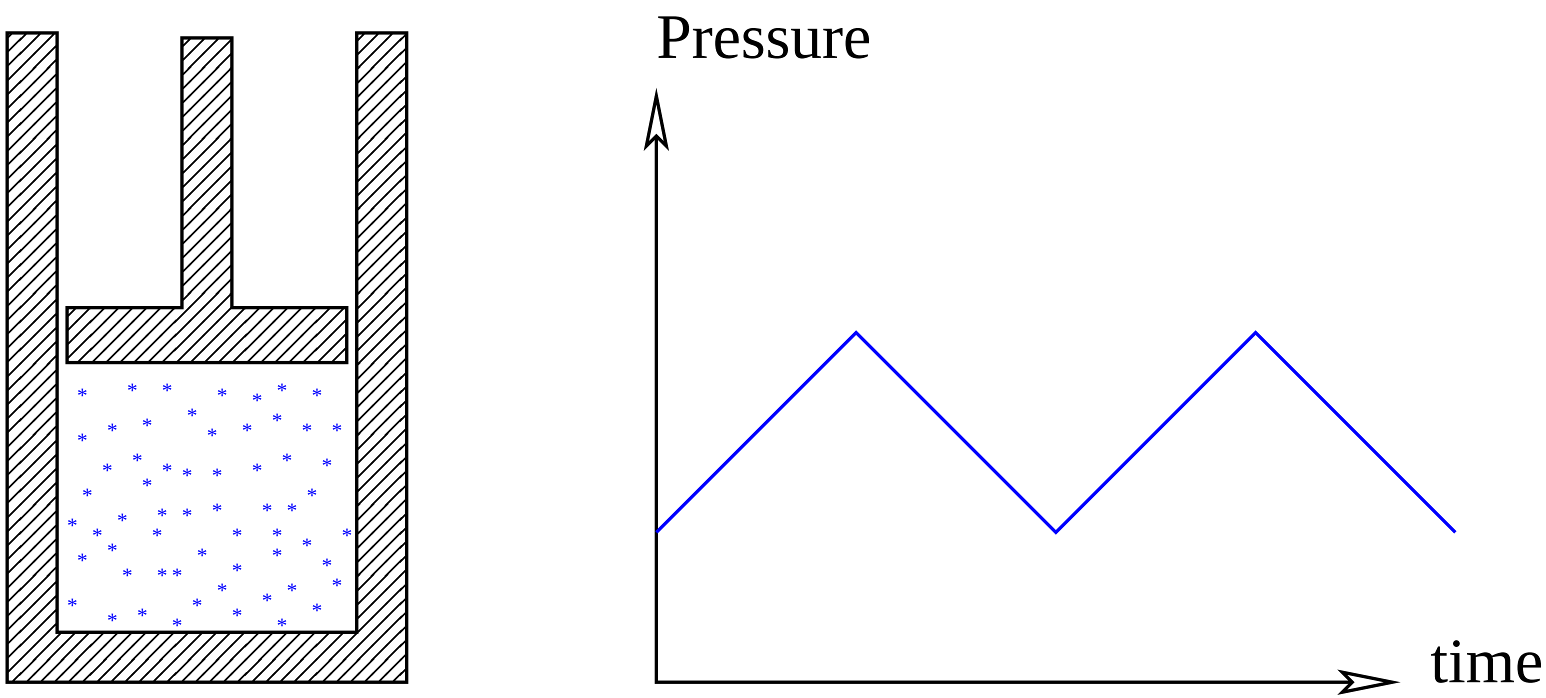
T^μ_μ spectral function and bulk viscosity
Where we can calculate it

Guy D. Moore, Omid Saremi

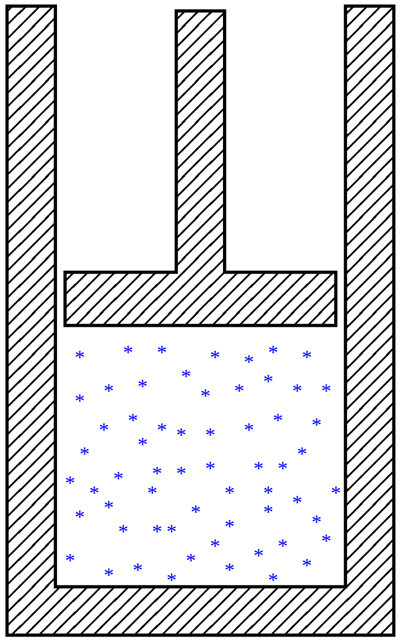
- Review of bulk viscosity, spectral function
- Perturbative regime: kinetic theory
  - High frequency: rising cut
  - Low frequency: peak
- Near the critical point: universal scaling
  - Dynamical universality classes: QCD vs. liquid-gas
  - Critical slowing down and Bulk viscosity
- Summary and conclusions

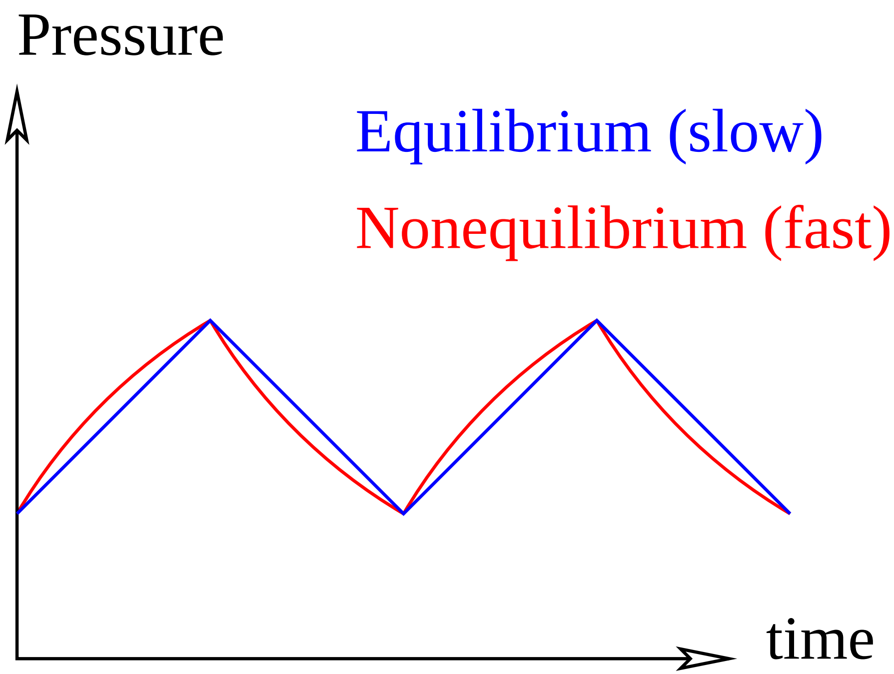
Raise and lower a piston: compress and decompress gas



Pressure rises and falls as you compress and decompress.

Compress faster: pressure deviates from equilibrium version



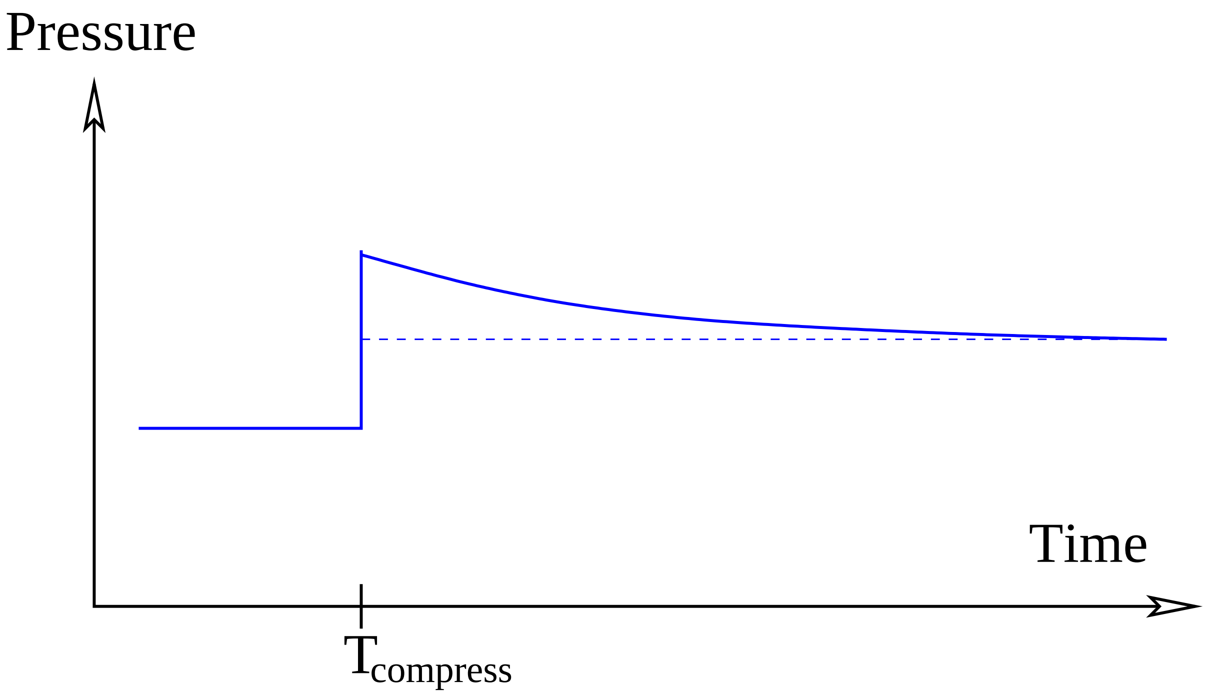


Compression: pressure higher

Decompression: pressure lower Second Law of Thermodynamics

Difference is characterized by Bulk Viscosity

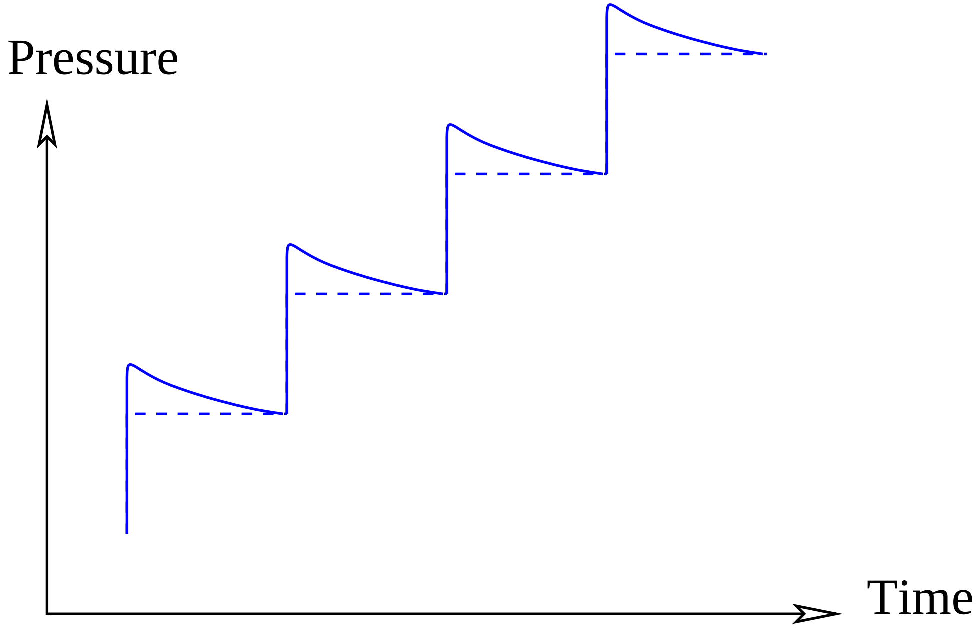
Consider small, sudden compression:



If operator $\mathcal{O}_1$ causes compression, $\mathcal{O}_2$ measures P:

- $\lim_{t\to 0}\langle \mathcal{O}_1(0)\mathcal{O}_2(t)\rangle$ gives height of discontinuity
- $\int_0^\infty dt\,\langle \mathcal{O}_1(0)\mathcal{O}_2(t)\rangle$ gives area under difference curve.
  Scale by $\Delta V/V$: that defines bulk visc ζ.

Think of steady compression as many small ones



Integrated extra pressure is

$$\int (P - P_{\rm eq})dt = (\Delta V_{\rm tot}) \int_0^\infty dt \, \langle \mathcal{O}_1(0)\mathcal{O}_2(t) \rangle$$

Interesting quantity is integrated extra pressure.

Defined as the bulk viscosity:

$$\int dt (P - P_{\rm eq}) = -\zeta \Delta V/V = -\zeta \int dt\, \vec{\nabla} \cdot \vec{v}$$

or

$$P - P_{\rm eq} = -\zeta\, \vec{\nabla} \cdot \vec{v}$$

Related to correlator of pressure operator $\mathcal{O}_2 = P = \frac{1}{3} T^i_i$ and gen. of expansions $\mathcal{O}_1 = \mathcal{O}_2$. Usual arguments:

$$\zeta = \frac{1}{2} \lim_{\omega \to 0} \frac{1}{\omega} \int_{-\infty}^{\infty} dt e^{i\omega t} \int d^3x \left\langle \frac{1}{9} \Big[T^i_i(x,t)\, ,\, T^j_j(0,0) \Big] \right\rangle \,.$$

And small t response described by ω integral.

Is it T^i_i? Or T^μ_μ?

It doesn't matter! $\Big[\mathcal{O}_1^\dagger\,,\,\mathcal{O}_1\Big] = \Big[\mathcal{O}_1^\dagger + c\,,\,\mathcal{O}_1 + c\Big]$.

T^0_0 acts almost like constant[a] as energy is conserved.

Useful choices:

- T^i_i: intuitively clear
- T^μ_μ: sum rules and exact results
- $T^i_i - \langle T^i_i\rangle \simeq T^i_i + 3c_s^2 T^0_0$: allows KMS

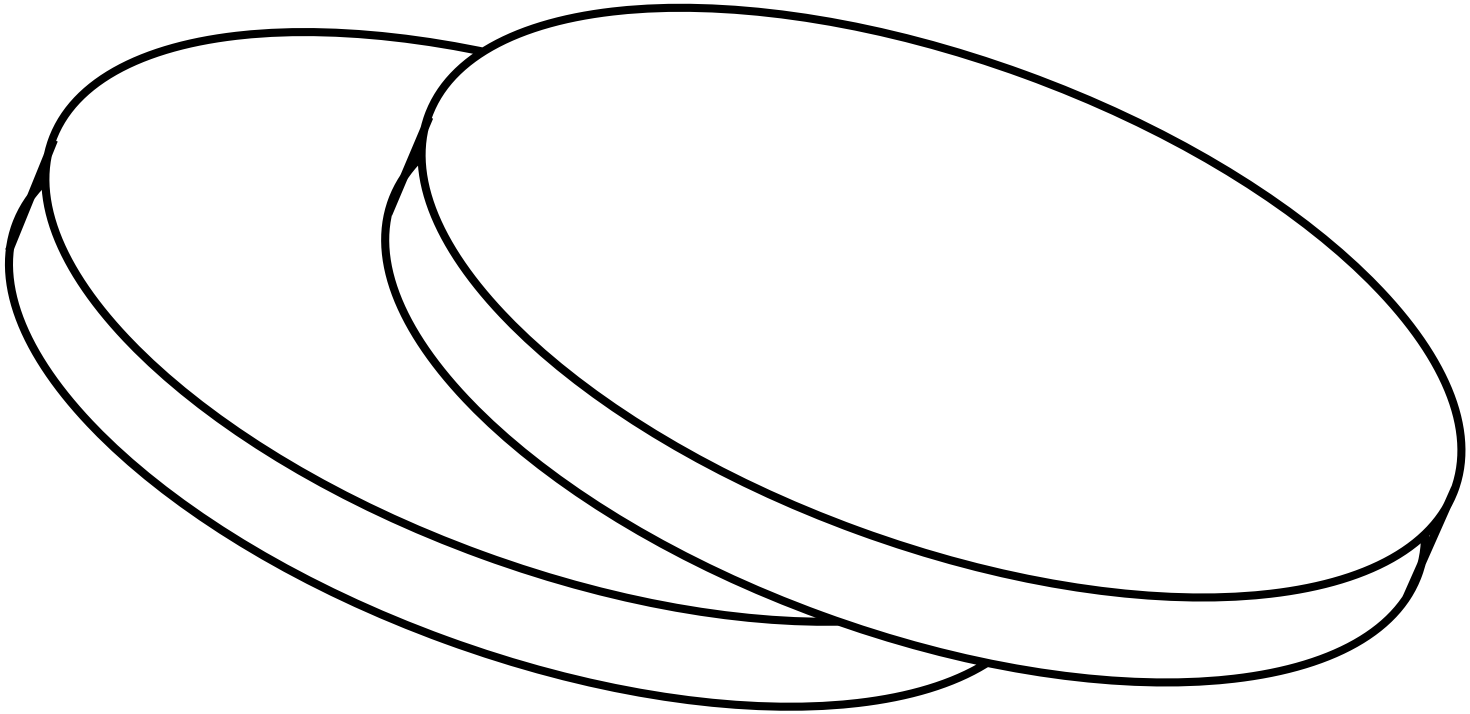
$$\int dt e^{i\omega t}\Big\langle \mathcal{O}^\dagger(t)\mathcal{O}(0)\Big\rangle = \frac{e^{\omega/T}}{e^{\omega/T}-1}\int dt e^{i\omega t}\Big\langle\Big[\mathcal{O}(t)\,,\,\mathcal{O}(0)\Big]\Big\rangle$$

without need to subtract disconnected part

[a] almost–T^{00} shift adds t-independent contrib: $\delta(\omega)$, const in $G_{\rm E}(\tau)$.

Application: heavy ion collisions

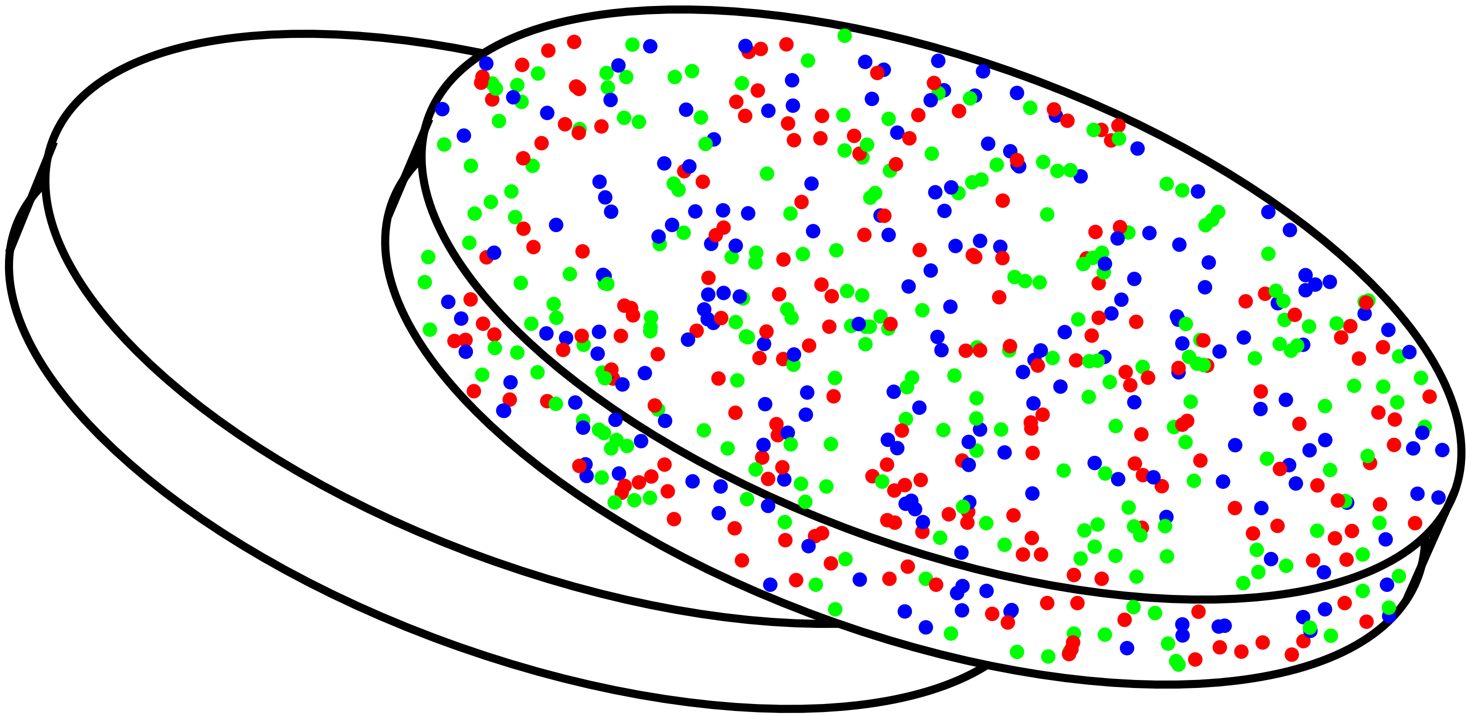
Accelerate two heavy nuclei to high energy, slam together.



Just before: Lorentz contracted nuclei

Heavy ion collisions

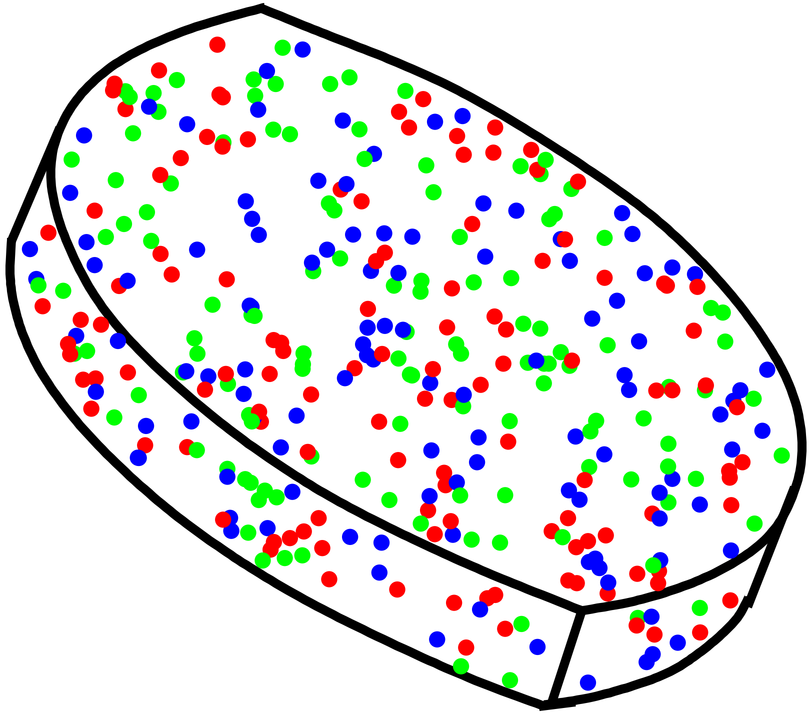
Each nucleus is $\sim 200\ p, n$, each built of $\sim 50\ q, \bar{q}, g$



It is the $q, \bar{q}, g$ which scatter.

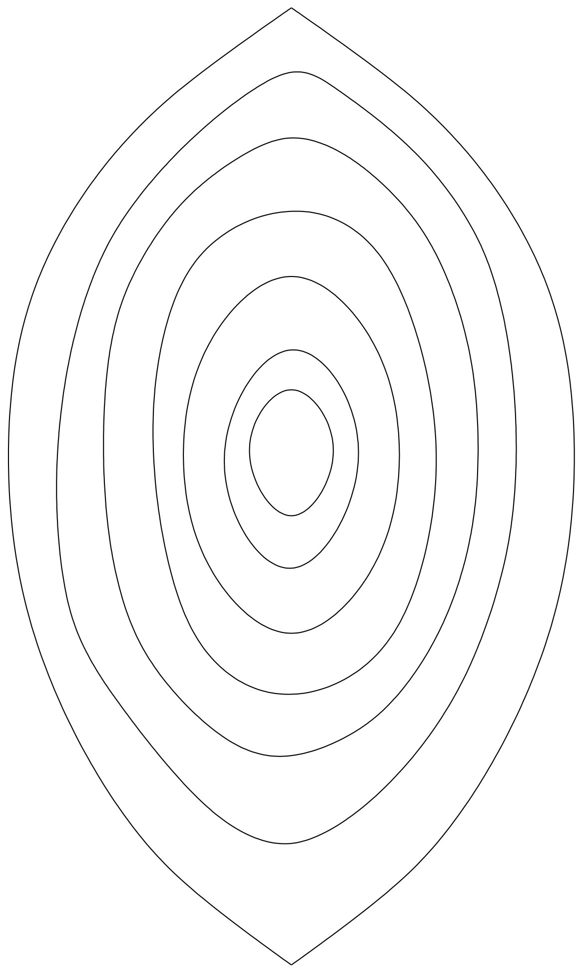
After the scattering:

"Flat almond" shaped region of $q, \bar{q}, g$ which scattered.



Few thousand. random velocities. Quark-Gluon Plasma

Density, pressure inhomogeneities



Ions "thicker" in centre. Hence, QGP also denser in centre. Outward press. gradients, larger transverse than longitudinal

Leads to radial expansion and to elliptic flow.

Radial expansion:

$$F(r) = \frac{dP(r)}{dr} = (\epsilon + P)\frac{dv(r)}{dt}$$

but reduction in effective P (bulk viscosity) lowers $v(r)$.

Similarly shear viscosity lowers elliptic flow.

One system: bulk viscosity "looks like" $P(\epsilon)$ (EOS)

Two systems, different sizes/shapes: can separate them.

Bulk viscosity can be measured

So let's calculate it!

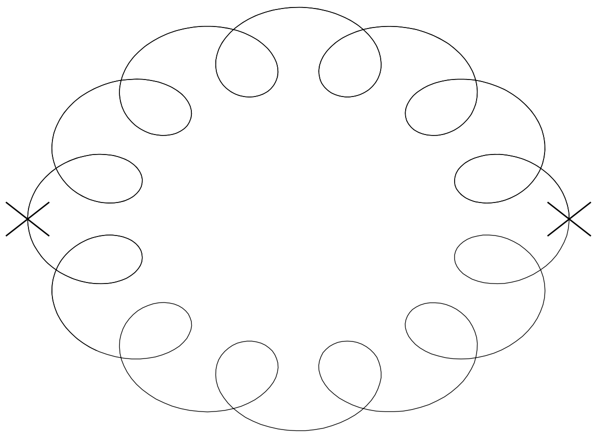
Perturbative regime

Normalize so $S = \int d^4x \frac{1}{2g^2} \mathrm{Tr}\, G_{\mu\nu} G^{\mu\nu}$.

Do pure glue for simplicity. Conformal anomaly:

$$T^\mu_\mu = \frac{\beta}{g^4} \mathrm{Tr}\, G_{\mu\nu} G^{\mu\nu}\,, \qquad \beta \equiv \frac{\mu^2 d}{d\mu^2} g^2 \sim g^4\,.$$

Evaluate Wightman correlator of $(\beta/g^4)G^2 - (1 - 3c_{\rm s}^2)T^0_0$.



Leading diagram.

Note $(1 - 3c_{\rm s}^2) \sim g^4$ is small;

$(1 - 3c_{\rm s}^2)T^0_0$ is g^2 suppressed.

Leading perturbative result

$$G^{>}(Q) = \frac{2\beta^2 d_{\rm A}}{9g^4}\int \frac{d^4P d^4R}{(2\pi)^8} G^{>}_{\mu\alpha}(P) G^{>}_{\nu\beta}(R) (2\pi)^4 \delta^4(Q-P-R)$$
$$\times (g^{\mu\nu} P{\cdot}R - P^\mu R^\nu)(g^{\alpha\beta} P{\cdot}R - P^\alpha R^\beta)$$

Cut propagator

$$G^{>}_{\mu\nu}(P) = [n_b(p^0)+1]\, 2\pi\delta(P^2+m_\infty^2) \sum_\lambda \epsilon_\mu(\lambda)\epsilon^*_\nu(\lambda)\,,$$

One contribution: both p^0, r^0 same sign.

$$G^{>}_{\rm cut}(\omega,0) = \left[n_b(\tfrac{\omega}{2}) + 1\right]^2 \frac{2\beta^2(g)}{9g^4}\;\frac{2d_{\rm A}\,\omega^4}{32\pi}$$

order $g^4\omega^4$ “cut” G_R has a cut, this is its discontinuity

“Pole” contribution

Other possibility: one line positive one negative frequency.

Naively: $P^2 = 0 = R^2$ and $P + R = 0$ so $P \cdot R = 0$. Get 0.

Less naive: $P^2 = -m_\infty^2 \sim g^2 T^2$.

Need $(1 - 3c_{\rm s}^2) T_0^0$ term (same order).

$$
\begin{aligned}
G^{>}_{\rm pole}(\omega, 0) \;\; &= \;\; \delta(\omega) \frac{2}{9} \, 2 d_{\rm A} \frac{1}{4\pi} \int_0^\infty n(p)(1 + n(p)) \\
&\qquad \times \left[\left(\frac{1}{3} - c_{\rm s}^2 \right) p^2 + \frac{\beta m_\infty^2}{g^2} \right]^2 dp \, .
\end{aligned}
$$

IR singular: Order $g^7 T^4$ area delta function at $\omega = 0$.

Need to know width of peak

Bulk viscosity is $G^{>}(\omega=0)/T$. Need width of peak

Include imaginary parts on propagators: need ladders as well

Amounts to kinetic treatment. T^{μ}_{μ} in terms of f:

$$(T^i_i + 3c_{\rm s}^2 T^0_0) = \sum \int \frac{d^3p}{(2\pi)^3}\left[\left(1-3c_{\rm s}^2\right)p^2 + \frac{3\beta m_\infty^2}{g^2}\right](f_0+\delta f)$$

Boltzmann equation

$$\mathbf{v}\cdot\boldsymbol{\nabla} f_0 + \partial_t f = -\mathcal{C}[f]$$

becomes

$$\frac{f_0(1{+}f_0}{ET}\left(\left[\frac{1}{3}-c_{\rm s}^2\right]p^2 - \frac{\beta m_\infty^2}{g^2}\right) = -i\omega\delta f - \mathcal{C}[f]\,.$$

Details of collisions do not change area of peak:

$$\delta f(\omega) = \frac{1}{\mathcal{C} - i\omega}[\text{source}] \quad \rightarrow \quad \int d\omega \delta f(\omega) = [\text{source}]$$

Shape of peak is crudely

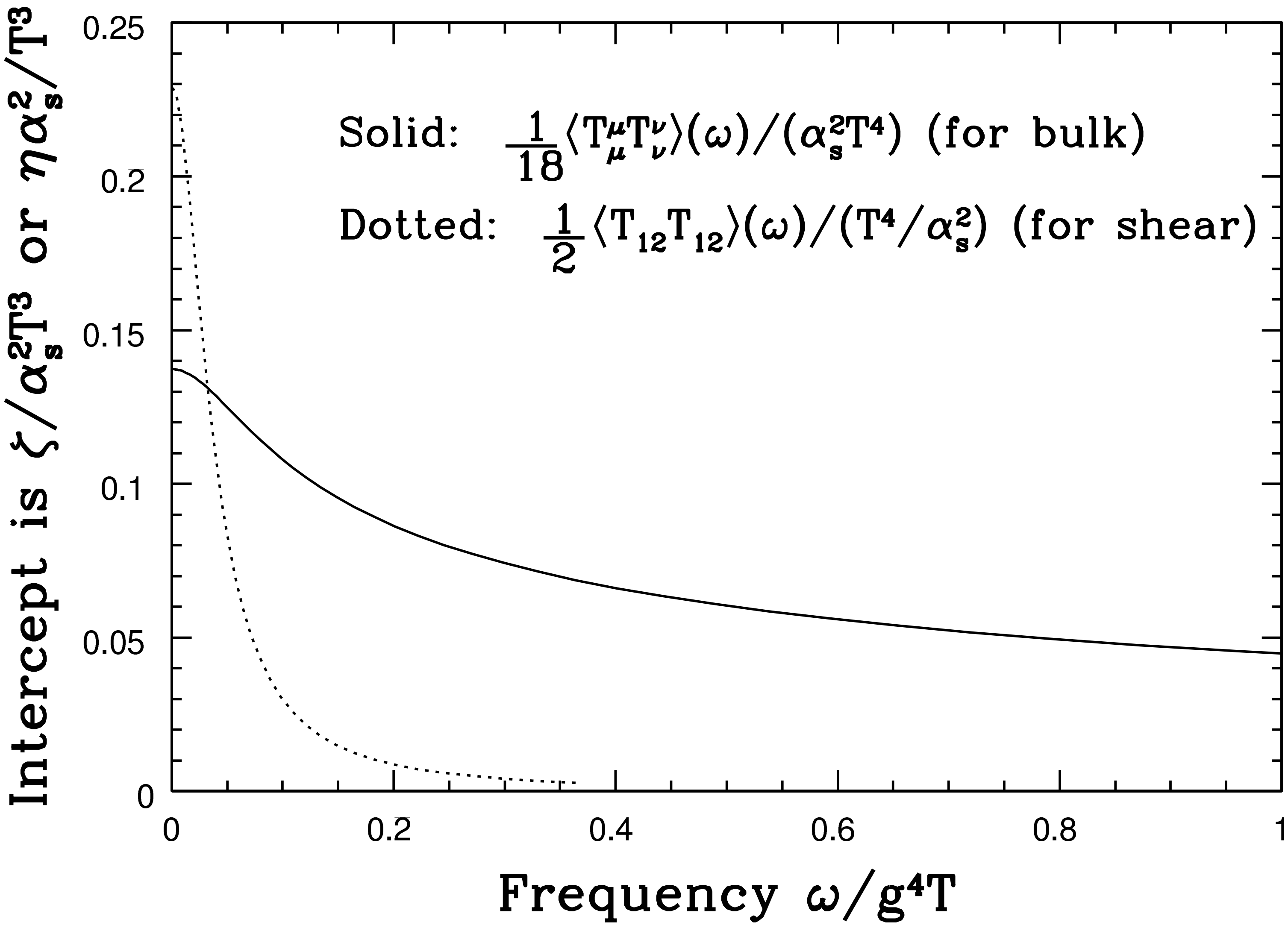
$$G^{>}(\omega) \sim \int \frac{d^3p}{(2\pi)^3} f_0[1+f_0] \left[\left(\frac{1}{3} - c_{\rm s}^2 \right) p^2 + \frac{\beta m_\infty^2}{g^2} \right]^2 \frac{\Gamma[p]}{\omega^2 + \Gamma^2[p]}$$

with $\Gamma[p] \sim g^4 T^3 / p^2$ the large-angle scatt. width

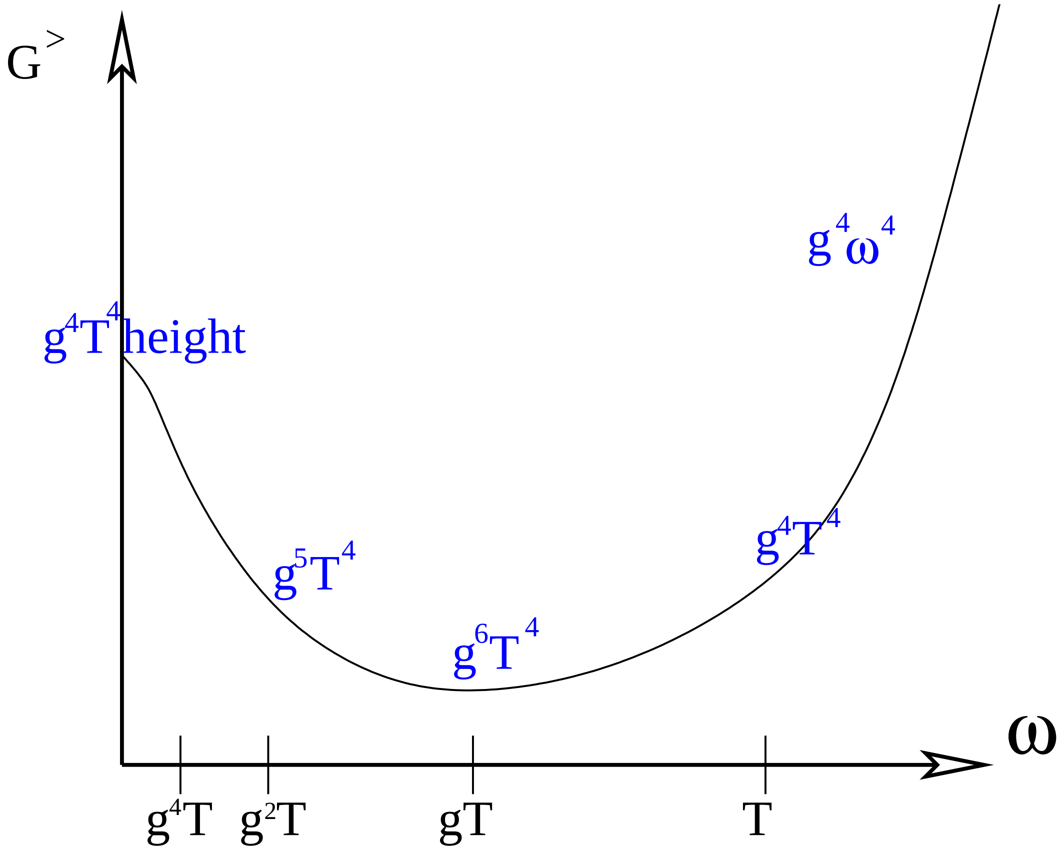
Peak height dominated by hard particles.

"Shoulders" and total area by soft particles.

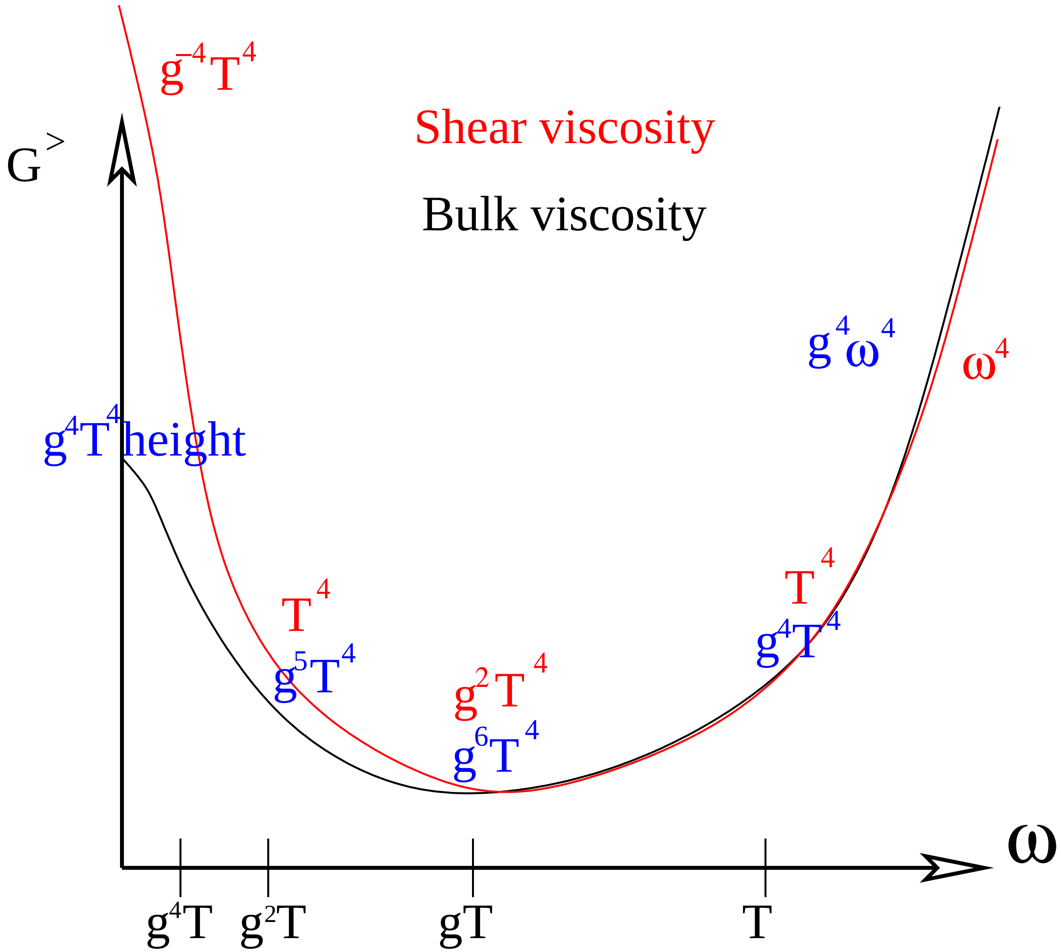
Shape of low frequency peak



Summary:



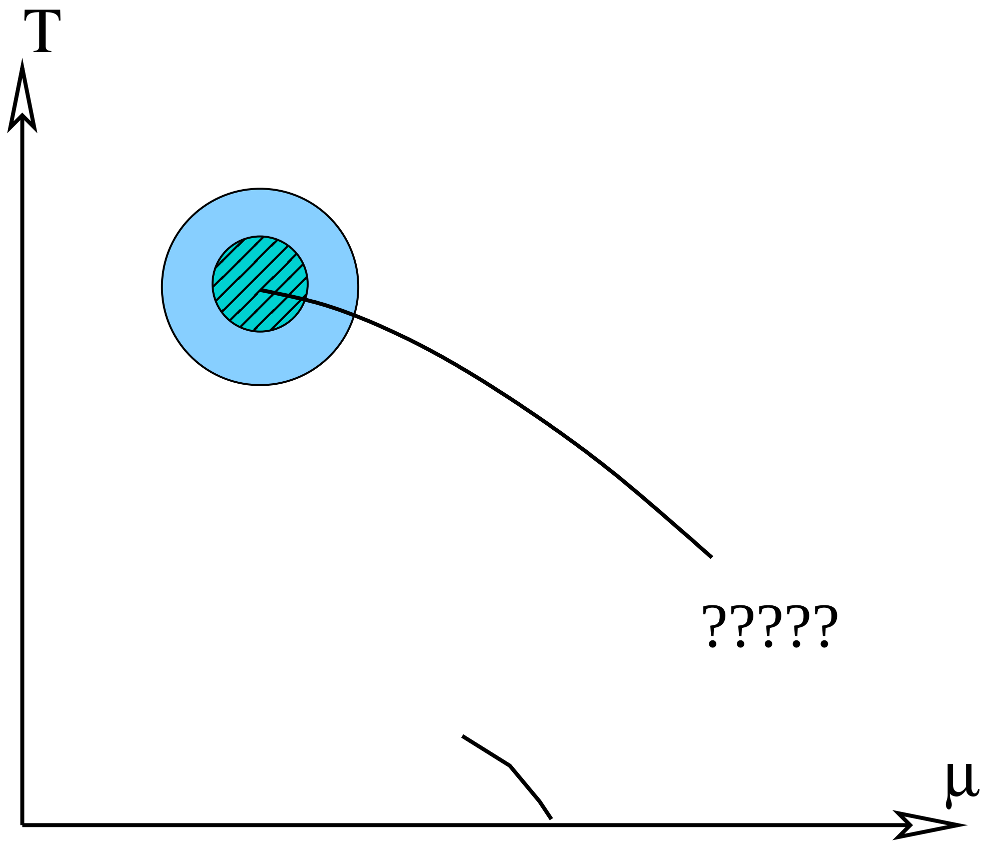
Summary:



g−4T4
Shear viscosity
Bulk viscosity
G>
g4ω4
ω4
g4T4 height
T4
g5T4
g2T4
g6T4
T4
g4T4
ω
g4T
g2T
gT
T

Another analytically tractible case

Critical region near second-order transition point:



Possible to compute parametric behaviors analytically

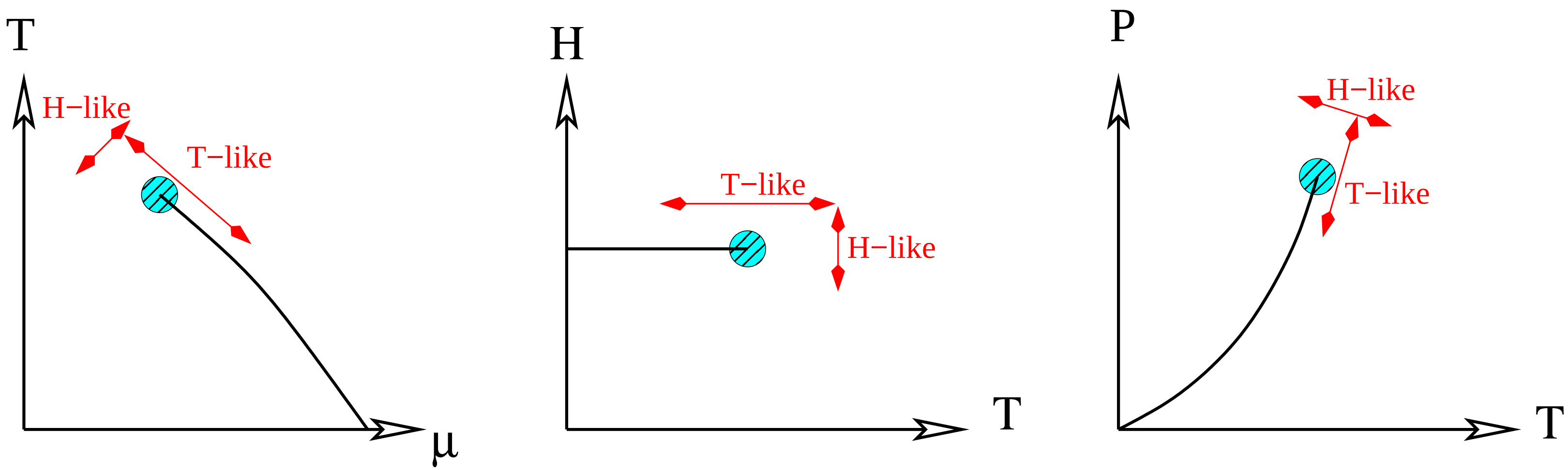
Static universality

"Chiral" phase transition not true symmetry breaking.

Order parameter $\boldsymbol{\psi} = \langle \bar{\psi}\psi \rangle$ same universality as Ising

Mapping of critical regions between

QCD (T, μ) Ising (H, T) Liq/Gas (P, T)



All display same static critical phenomena.

Statics

Distance from critical point $t \equiv |T - T_c|/T_c$.

Order parameter ψ correl length $\xi \sim t^{-\nu}$ $[\nu \simeq 0.630]$

Free energy (pressure) finite, nonsingular:

$$F[T] = F_{\text{nonsing}}[T] + t^{2-\alpha} F_{\text{sing}}$$

with $\alpha = 0.110$. Energy nonsingular

$$E = T\frac{\partial F}{\partial T} + F = E_{\text{nonsing}} + t^{1-\alpha}(2-\alpha)F_{\text{sing}}$$

but heat capacity IS singular:

$$C_v \sim \frac{\partial E}{\partial T} \sim t^{-\alpha} + C_{v,\text{nonsing}}$$

Dynamics

Two sets of degrees of freedom:

- Short-distance: $\Delta P/\Delta E \sim 1$. Rapid equilibration
- Long-range [ψ fluct]: $\Delta P/\Delta E = 0$. Dominate C_v.

Long range equilibrate diffusively and slowly:

$$\langle \boldsymbol{\psi}(k,t)\boldsymbol{\psi}(-k,0)\rangle \sim \chi(k)\ \exp(-t/\tau)\,, \qquad \tau \sim k^{-z}$$

with z dynamic critical exponent.

Sudden compression:

- First, short-dist DOF adjust, $\Delta P \sim \Delta E$
- Later, ψ adjusts, ΔP relaxes back in time $\propto \xi^{-z}$

Dynamic universality: value of z

Long scale dynamics essentially hydrodynamic
Hohenberg Halperin Rev Mod Phys 49 p435 (1977)

Depend on what quantities are conserved (ψ is not)
Conserved: $T^{0\mu} = (\epsilon, \vec{P})$ and ρ_B
Liquid-gas system: $\epsilon, \vec{P}, \rho$ conserved.

Same dynamic universality as Liquid-Gas

Son Stephanov hep-ph/0401052

Dynamics analyzed to death by CM physics people: $z \simeq 3$.

Even bulk viscosity analyzed Onuki, Phys Rev E 55 p403 (1997)

Previous argument: $\Delta V/V$ leads to instantaneous $\Delta P \sim \Delta E \sim T^4 \Delta V/V$.

But ΔP relaxes to $\simeq 0$ in time $\tau \sim \xi^z$.

Hence, expect $\zeta \sim T^3 (\xi T)^z \sim T^3 t^{-z\nu} \simeq T^3 t^{-2}$.

Subtlety: $\boldsymbol{\psi}$ relaxes in stages from $k \sim T$ to $k \sim \xi^{-1}$.

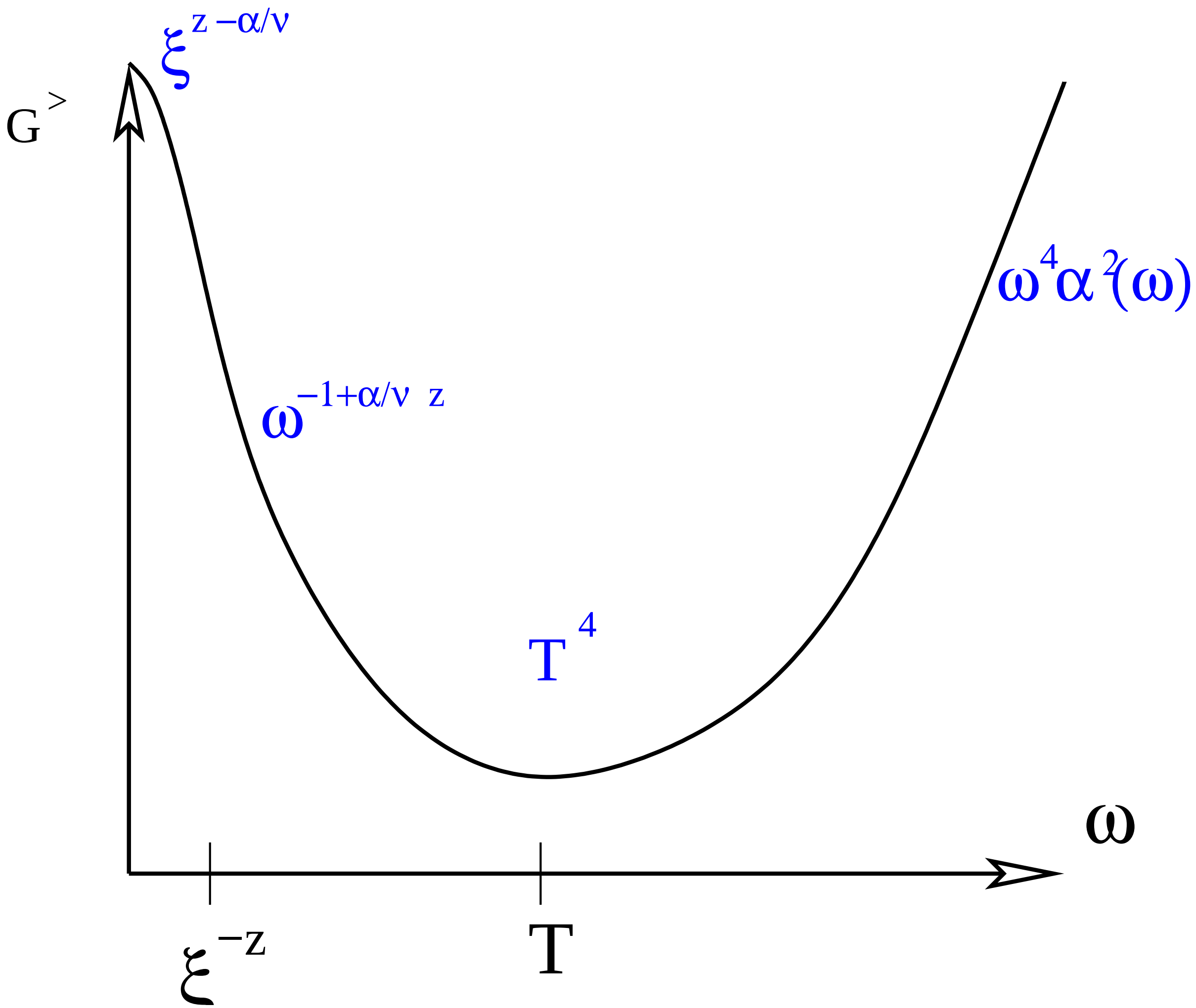
At time τ, modes with $k > \tau^{-1/z}$ are equilibrated.

These carry $C_v \sim k^{-\alpha/\nu}$. So $\Delta P \sim \Delta E k^{\alpha/\nu}$.

Integrate this behavior for all intermediate times....

$$\zeta \sim T^3 (\xi T)^{z-\alpha/\nu} \sim T^3 t^{-z\nu+\alpha}$$

Slow dynamics: another low ω peak!



Implications for Euclidean correlators

Integral relation between $G_{\rm E}(\tau)$ and ρ:

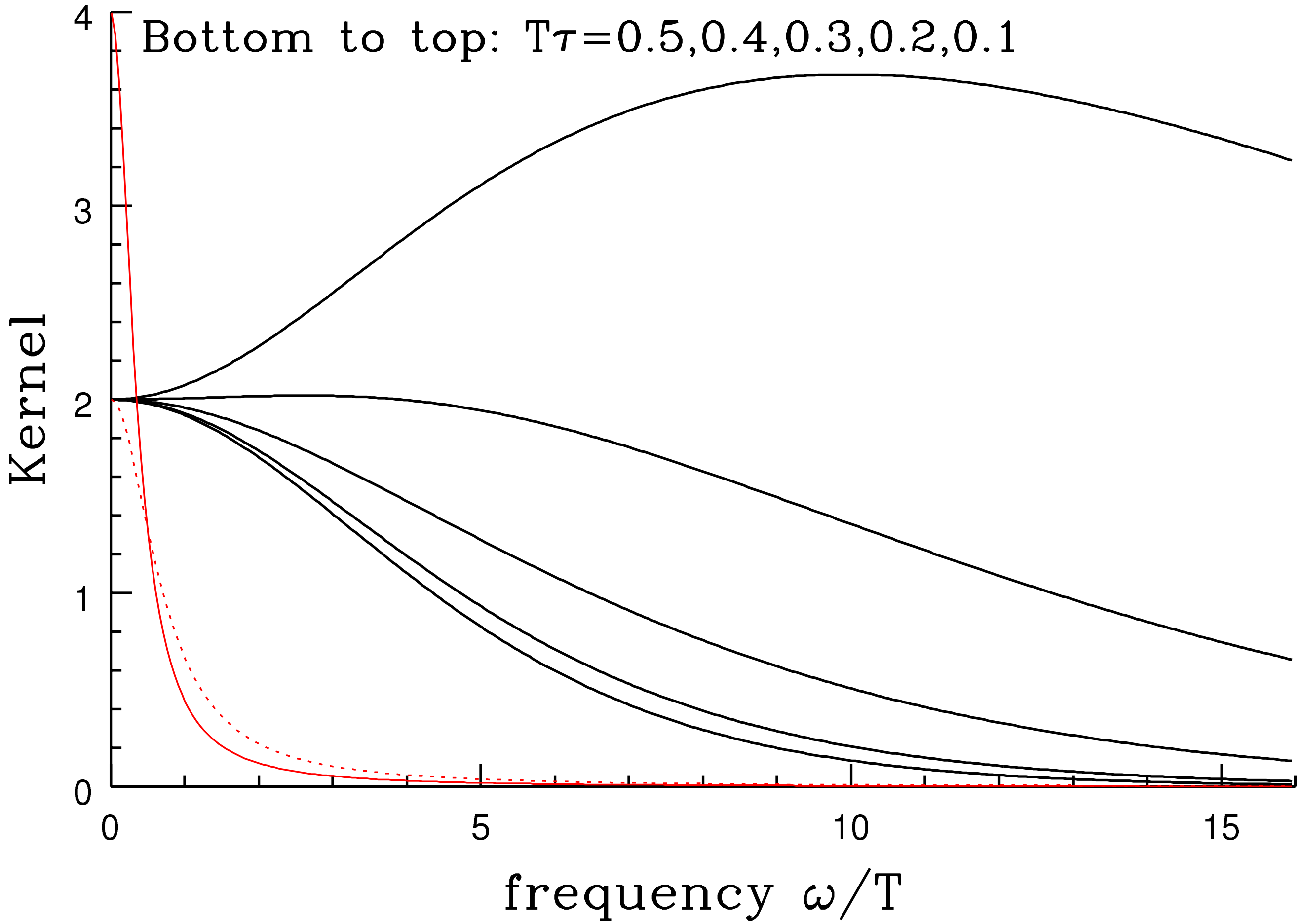
$$G_{\rm E}(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\left\langle \left[\mathcal{O}_1 \, , \, \mathcal{O}_1 \right] \right\rangle (\omega)}{\omega} K(\omega,\tau) \, ,$$

$$K(\omega,\tau) = \frac{\omega \cosh[\omega(\tau - \beta/2)]}{\sinh(\beta\omega/2)} \, .$$

Knowing ρ, can compute $G_{\rm E}$.

Then, ask how we could "guess" that ρ from $G_{\rm E}$.

The function $K(\omega, \tau)$



Peak near zero gives common contrib to $G_E(\tau)$, all τ

Implications for spectral function

Sharp peak: contribution to $G_{\mathrm{E}}(\tau)$

- Almost independent of shape of peak
- Almost independent of value of τ

All $G_{\mathrm{E}}(\tau)$ raised by common amount: Area under peak.

- Determined by static universality
- Does not diverge as $T \to T_c$

Shape of peak essential to finding ζ =height.
Very hard to determine from $G_{\mathrm{E}}(\tau)$.

Conclusions

- Slow equilibration $\Rightarrow$ Peak in spectral function
- Slow equil. at weak coupling: "Wide shouldered" peak
- Slow equilibration near critical point:
  nearly ω^{-1} shaped peak, $\zeta \sim \xi^{z-\alpha/\nu}$
- Euclidean Green function cannot find shape of peak

If crossover is rapid (near-critical): big ζ

In this case, Euclid. methods CANNOT say much directly

we CAN say something useful from universality+static info

Comment on Kharzeev and Tuchin

Their sum rule:

$$G_{\rm E}(\omega'=0) = \int d\omega \frac{\rho}{\omega} = \frac{T^5 \partial}{\partial T}\left(\frac{\epsilon - 3P}{T^4}\right)$$

$g \ll 1$: RHS$\sim g^6 T^4$. LHS$\sim g^7 T^4$[peak]$+ g^4 T^4$[cut]

Similar sum rule for $T_{xy}T_{xy}$ correlator

$$G_{\rm E,T_{xy}T_{xy}}(\omega'=0) = \int d\omega \frac{\rho}{\omega} = P$$

Making same assumption about *shape* of peak as they make gives $\eta \sim T^3/g$ not T^3/g^4.

Also, convergence/validity questions with Kramers-Kronig