Assignment 1

Submit by: 12th September

1. The three coordinate base vectors of an oblique coordinate system are

$$\begin{aligned} \mathbf{e_1} &= \mathbf{i} \\ \mathbf{e_2} &= 0.4\,\mathbf{i} + \mathbf{j} \\ \mathbf{e_3} &= 0.2\,\mathbf{i} + 0.3\,\mathbf{j} + \mathbf{k} \end{aligned}$$

   where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form the standard Cartesian basis.

   a) Write down the transformation matrix for the components of a vector, V^i, from the Cartesian to the $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ system.
   b) Calculate g_{ij} and Γ^k_{ij} for this system.
   c) Now consider the coordinate system given by the base vectors

$$\begin{aligned} \mathbf{u_1} &= \frac{\mathbf{e_2} \times \mathbf{e_3}}{\mathbf{e_1} \cdot \mathbf{e_2} \times \mathbf{e_3}} \\ \mathbf{u_2} &= \frac{\mathbf{e_3} \times \mathbf{e_1}}{\mathbf{e_1} \cdot \mathbf{e_2} \times \mathbf{e_3}} \\ \mathbf{u_3} &= \frac{\mathbf{e_1} \times \mathbf{e_2}}{\mathbf{e_1} \cdot \mathbf{e_2} \times \mathbf{e_3}} \end{aligned}$$

   Show that the transformation matrix of V^i to this system is identical to the transformation matrix of V_i to the $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ system.

2. a) Show that the covariant derivatives of the metric tensor vanish:

$$D_\lambda g_{\mu\nu} = D_\lambda g^{\mu\nu} = 0.$$

   b) Show that $g^{-1/2}\epsilon^{\mu\nu\rho\sigma}$ transforms like a tensor under a general coordinate transformation, where $g = -\det(g_{\mu\nu})$, $\epsilon^{\mu\nu\rho\sigma}$ is completely antisymmetric in μ, ν, ρ, σ and $\epsilon^{0123} = 1$. Show that its covariant derivative vanishes.

3. Write down Laplace's equation, $\nabla^2\,\Psi(\vec{r}) = 0$, in the coordinate system (μ, ν, ϕ), defined as

$$
\begin{aligned}
x &= a\ \cosh(\mu)\ \cos(\nu)\ \cos(\phi) \\
y &= a\ \cosh(\mu)\ \cos(\nu)\ \sin(\phi) \\
z &= a\ \sinh(\mu)\ \sin(\nu)
\end{aligned}
$$

where μ is a nonnegative real number, $\nu \in [-\pi/2, \pi/2]$ and $\phi \in (-\pi, \pi]$. Writing $\Psi(\vec{r}) = U(\mu)\ V(\nu)\ \Phi(\phi)$, show that the Laplace's equation can be separated into three ordinary differential equations in μ, ν, ϕ.

4. Use the principles of general covariance to write the equation of motion of a free particle in a uniformly rotating frame (rotating about a fixed axis with a constant angular velocity). Interpret the different terms in the equation.

Marks: 10+10+10+20