Assignment 3

Submit by: 26th October

1. Evaluate $\lim_{t\to\infty} x_2(t)$, where $x_2(t)$ is defined by the system

$$\begin{aligned} x_1 \,+\, t\,x_2 \,+\, t^2\,x_3 &= t^4, \\ t^2\,x_1 \,+\, x_2 \,+\, t\,x_3 &= t^3, \\ t\,x_1 \,+\, t^2\,x_2 \,+\, x_3 &= 0. \end{aligned}$$

2. Let $f_1(x)$, ..., $f_n(x)$ be n functions whose all derivatives upto order (n-1) are defined. The determinant

$$\begin{vmatrix} f_1(x) & f_2(x) & ... & f_n(x) \\ f_1'(x) & f_2'(x) & ... & f_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & ... & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the Wronskian, $\mathcal{W}[f_1(x), ..., f_n(x)]$.

a) If the functions f_1 ... f_n are linearly dependent, show that $\mathcal{W}$ must vanish everywhere.
b) Show that

$$\mathcal{W}[g(x)f_1(x),\ ...,\ g(x)f_n(x)] \;=\; (g(x))^n\ \mathcal{W}[f_1(x),\ ...,\ f_n(x)].$$

3. Find all the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_4 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_1 \end{pmatrix}$$

where a_1, a_2, a_3, a_4 are real numbers.

4. A hermitian matrix H is called positive definite if for any nonzero vector $|x\rangle$, $\langle x|H|x\rangle > 0$.
a) Show that H is positive definite if and only if its eigenvalues are all positive.

b) For any nonsingular square matrix A, show that $AA^\dagger$ is positive definite.
c) Show that such a matrix can be decomposed as $A = HU$, where H is positive definite hermitian and U is unitary.
d) Therefore show that A can be "diagonalized" by a biunitary transformation: $A = S^\dagger MU$ where M is diagonal, with positive entries only, and S, U are unitary matrices.

5. a) Find the Jordan canonical form of the matrix

$$A = \begin{pmatrix} 6 & 5 & 2 \\ -7 & -6 & -3 \\ 10 & 11 & 6. \end{pmatrix}$$

Now find the similarity transformation which takes A to the Jordan form.

(Do not use JordanDecomposition in Mathematica for this problem.)

b) Using the solution to the above part, solve for the initial value problem

$$\begin{aligned} \dot{x} &= 6x + 5y + 2z \\ \dot{y} &= -7x - 6y - 3z \\ \dot{z} &= 10x + 11y + 6z. \end{aligned}$$

Here x, y, z are functions of time, $\dot{x}$ refers to time derivative of x, and at $t = 0$, $x = 1$, $y = -2$, $z = 3$.

6. Find the group of symmetries of the square (called C_{4v}).
Find its conjugacy classes.
What are the dimensionalities of its irreps?
Construct the character table of the group.

Marks: 5+10+10+15+15+10