Assignment 4

Submit by: 23rd November

1. Consider matrices of the form $\Sigma = a_0 I + i\sum_{i=1}^{3} a_i\sigma_i$, where I is the 2×2 identity matrix, σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and a_0, a_i are real numbers satisfying $a_0^2 + \sum_{i=1}^{3} a_i^2 = 1$.

a) Show that the matrices Σ are unitary, and have determinant $= 1$.

b) Show that the set of all Σ form a group.

c) Is this group simple/semisimple?

d) Show that the matrices $\Sigma' = a_0 I + i a_1 \sigma_1$, $a_0^2 + a_1^2 = 1$, form an Abelian subgroup of this group.

2. a) Find the power series expansion of $f(z) = 1/(1+z^2)$ around the point z=1. What is the radius of convergence of the series?

b) Find the Laurent series for $1/z^2(1-z)$ in the region $0 < |z| < 1$.

c) Complete the proof of Laurent series expansion by using the expansion ($|z| < 1$)

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^{n-1} + R_n, \qquad R_n = \frac{z^n}{1-z}$$

in the integrals over the circles C_1, C_2, and show that as $n \to \infty$ the contributions coming from R_n go to zero.

3. Evaluate the integrals

a) $\int_C \; z^2 \exp(z)\, dz$, where C is the closed triangle with vertices at $z = 1, -1$ and $2i$.

b) $\int_C \; z \exp(z^2)\, dz$, where C is the path from the origin to the point 1+i along the parabola $y = x^2$.

c) $\int_C \; z^*\, dz$ where C is the circle given by $|z+1| \; = \; 2$.

4. Calculate

$$a) \quad \int_0^\infty \frac{(\log x)^2}{1+x^2}\, dx;$$

$$b) \quad \int_0^\infty \sin(x^2)\, dx;$$

$$c) \quad \int_0^{2\pi} \frac{d\theta}{1 - 2t\cos\theta + t^2}, \qquad |t| < 1.$$

5. Use contour integration techniques to evaluate the series

$$a) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}; \qquad b) \sum_{n=-\infty}^{\infty} \frac{1}{a^4 + n^4}.$$

Use the result of b) to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

6. Discuss the singularity structure and the Riemann surfaces for the functions

a) $\sqrt{(z-1)(z^2+1)}$, and

b) $\ln(2z+i)$.

Marks: 10+15+10+15+15+10