①

# Final Exam., Mathematical Methods Course

24th November, 2 p.m. – 5 p.m.

1. a) An $n \times n$ matrix $A$ satisfies a polynomial equation

$$p(A) = c_0 I_n + c_1 A + c_2 A^2 + c_3 A^3 + ... = 0,$$

where $I_n$ is the $n \times n$ identity matrix. If $B$ is connected to $A$ via a similarity transformation, $B = S^{-1}AS$, show that $p(B) = 0$.

**6**

b) If $n = 2$, argue that $\exp(A)$ can be written as a linear combination of $I_2$ and $A$ (you may use Cayley-Hamilton theorem). In this case, if the eigenvalues of $A$ are $x_1$ and $x_2$ and $x_1 \neq x_2$, show that

$$\exp(A) = \frac{x_1 e^{x_2} - x_2 e^{x_1}}{x_1 - x_2} I_2 + \frac{e^{x_1} - e^{x_2}}{x_1 - x_2} A.$$

What about the case when $x_1 = x_2$?

**7+2**

(a) If $B = S^{-1}AS$, $B^n = S^{-1}AS . S^{-1}AS . \dots . S^{-1}AS = S^{-1}A^n S$

Using $I_n = S^{-1} I_n S$, $\Rightarrow p(B) = S^{-1}(c_0 I_n + c_1 A + c_2 A^2 + \cdots) S = S^{-1}p(A)S = 0$

b) Since a $n=2$ matrix $A$ satisfies $(A - x_1 I_2)(A - x_2 I_2) = 0$ if $x_1, x_2$ are its eigenvalues (Cayley-Hamilton theorem), $\Rightarrow A^2 = (x_1 + x_2)A - x_1 x_2 I_2$ $\longrightarrow$ ①

Therefore any power $A^n$, $n \geq 2$, can be written as $c_{1,n} A + c_{2,n} I_2$

$\Rightarrow e^A = \sum_n \frac{1}{n!} A^n$ can be written as $\lambda_1 A + \lambda_2 I_2$

If $A$ has non-degenerate eigenvalues, by a similarity transformation we can diagonalize $A$: $B = S^{-1}AS = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$



$\Rightarrow$ By similar argument as in a), $e^B = \lambda_1 B + \lambda_2 I_2$

$$\Rightarrow \begin{pmatrix} e^{x_1} & 0 \\ 0 & e^{x_2} \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 + \lambda_2 & 0 \\ 0 & \lambda_1 x_2 + \lambda_2 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} \lambda_1 x_1 + \lambda_2 &= e^{x_1} \\ \lambda_1 x_2 + \lambda_2 &= e^{x_2} \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda_1 &= \frac{e^{x_1} - e^{x_2}}{x_1 - x_2} \\ \lambda_2 &= e^{x_1} - x_1 \frac{e^{x_1} - e^{x_2}}{x_1 - x_2} \\ &= \frac{x_1 e^{x_2} - x_2 e^{x_1}}{x_1 - x_2} \end{aligned}$$

Now, say, $x_2 = x$, $x_1 = x + \varepsilon$, and we want to take $\varepsilon \to 0$

$$\Rightarrow \lambda_1 \to \lim_{\varepsilon \to 0} \frac{e^{x}(e^{\varepsilon} - 1)}{\varepsilon} = e^{x},$$

$$\lambda_2 = \lim_{\varepsilon \to 0} \frac{(x+\varepsilon) e^{x} - x e^{x+\varepsilon}}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{(x+\varepsilon) e^{x} - x e^{x}(1+\varepsilon)}{\varepsilon}$$

$$= (1-x) e^{x}.$$

$$\Rightarrow e^{A} = e^{x} A + (1-x) e^{x} I_2 \longrightarrow (2)$$

This result is trivially correct when $A$ is diagonalizable, for then

$A = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = x I_2$ in any basis and $e^{A} = e^{x} I_2$.

What if $A$ is not diagonalizable? Then limit $\varepsilon \to 0$ from the non-degenerate case is non-trivial: $S$ becomes ill-defined as $\varepsilon \to 0$. Is (2) still true?

In this case, writing $e^{A} = \lambda_1 A + \lambda_2 I_2$, we can transform to the Jordan form $B = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$ and $e^{B} = \lambda_1 B + \lambda_2 I_2 \longrightarrow (3)$

Now $B^2 = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} = \begin{pmatrix} x^2 & 2x \\ 0 & x^2 \end{pmatrix}$

$$B^3 = \begin{pmatrix} x^2 & 2x \\ 0 & x^2 \end{pmatrix}\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} = \begin{pmatrix} x^3 & 3x^2 \\ 0 & x^3 \end{pmatrix}$$

and, if $B^{n-1} = \begin{pmatrix} x^{n-1} & (n-1)x^{n-2} \\ 0 & x^{n-1} \end{pmatrix}$,

$$B^n = B^{n-1} \cdot \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} = \begin{pmatrix} x^n & n x^{n-1} \\ 0 & x^n \end{pmatrix}$$

$$\Rightarrow e^B = \sum_{n=0}^{\infty} \frac{1}{n!} B^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n=1}^{\infty} \frac{1}{n!} \begin{pmatrix} x^n & n x^{n-1} \\ 0 & x^n \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \sum_{n=1}^{\infty} \frac{1}{n!} x^n & \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} \\ 0 & 1 + \sum_{n=1}^{\infty} \frac{1}{n!} x^n \end{pmatrix} = \begin{pmatrix} e^x & e^x \\ 0 & e^x \end{pmatrix}$$

From (3) then, $\begin{pmatrix} e^x & e^x \\ 0 & e^x \end{pmatrix} = \begin{pmatrix} \lambda_1 x + \lambda_2 & \lambda_1 \\ 0 & \lambda_1 x + \lambda_2 \end{pmatrix}$

$$\Rightarrow \lambda_1 = e^x, \quad \lambda_2 = (1-x)e^x \quad \Rightarrow \text{eqn. (2)}.$$

**Notes** a) In part a), many of you have used arguments like if $p(A) = 0$, $p(A)$ is the characteristic polynomial of $A$ and $p(\lambda) = 0$. Since $B = S^{-1}AS$ has same eigenvalues, it has same characteristic eqn. and therefore $p(B) = 0$. This is not correct because I never said $p(A)$ is the characteristic polynomial (defined by the polynomial $p(x) = \det|A - xI|$).

A matrix can satisfy $p(A) = 0$ though $p(A)$ is not the characteristic polynomial. [Can you give a simple example?]

Also, if the eigenvalues $\lambda$ of a matrix $B$ satisfy $p(\lambda) = 0$, that does not imply $p(B) = 0$ [unless $p(\lambda)$ is the characteristic polynomial $\det|B - \lambda I|$].

b) Some of you started with eqn. ① and calculated $e^A$ directly. Though the algebra is a bit longer, that is perfectly all-right if you can manage to get a general form of $A^n$ (One of you managed to do it).

From (1): 
$$A^2 = (x_1+x_2)A - x_1x_2 I_2 = \frac{x_1^2 - x_2^2}{x_1 - x_2} A - \frac{x_1x_2(x_1 - x_2)}{x_1 - x_2} I_2$$

$$A^3 = \frac{x_1^2 - x_2^2}{x_1 - x_2}\left((x_1+x_2)A - x_1x_2 I_2\right) - \frac{x_1x_2(x_1-x_2)}{x_1-x_2} A$$

$$= \frac{x_1^3 - x_2^3}{x_1 - x_2} A - \frac{x_1x_2(x_1^2 - x_2^2)}{x_1 - x_2} I_2$$

If $A^{n-1} = \frac{x_1^{n-1} - x_2^{n-1}}{x_1 - x_2} A - x_1x_2 \frac{x_1^{n-2} - x_2^{n-2}}{x_1 - x_2} I_2$,

$$A^n = \frac{x_1^{n-1} - x_2^{n-1}}{x_1 - x_2}\left((x_1+x_2)A - x_1x_2 I_2\right) - \frac{x_1x_2}{x_1-x_2}\left(x_1^{n-2} - x_2^{n-2}\right) A$$

$$= \frac{x_1^n - x_2^n}{x_1 - x_2} A - \frac{x_1x_2}{x_1 - x_2}\left(x_1^{n-1} - x_2^{n-1}\right) I_2$$

$$\Rightarrow e^{A} = \sum_{n=0}^{\infty} \frac{1}{n!} A^{n} = I_2 + \sum_{n=1}^{\infty} \frac{1}{n!} A^{n}$$

$$= I_2 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{x_1^{n} - x_2^{n}}{x_1 - x_2} A - I_2 \sum_{n=1}^{\infty} \frac{1}{n!} \left(x_1^{n-1} - x_2^{n-1}\right) \frac{x_1 x_2}{x_1 - x_2}$$

$$= \frac{e^{x_1} - e^{x_2}}{x_1 - x_2} A + I_2 \left[ 1 - \frac{x_1 x_2}{x_1 - x_2} \sum_{n=1}^{\infty} \frac{1}{n!} \left(x_1^{n-1} - x_2^{n-1}\right) \right]$$

$$[\quad] = 1 - \frac{1}{x_1 - x_2} \left( \sum_{n=1}^{\infty} \frac{1}{n!} x_1^{n} x_2 - \sum_{n=1}^{\infty} \frac{1}{n!} x_2^{n} x_1 \right)$$

$$= 1 - \frac{1}{x_1 - x_2} \left[ \left(e^{x_1} - 1\right) x_2 - \left(e^{x_2} - 1\right) x_1 \right]$$

$$= \frac{x_2 e^{x_1} - x_1 e^{x_2}}{x_1 - x_2}$$

$$\Rightarrow e^{A} = \frac{e^{x_1} - e^{x_2}}{x_1 - x_2} A + \frac{x_2 e^{x_1} - x_1 e^{x_2}}{x_1 - x_2} I_2$$

The $x_1 = x_2$ case can be done either using ~~l'Hospital~~ the $\varepsilon \to 0$ limit, or starting from (2) and using similar manipulations as above.

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2. By calculating the secular determinant, find the eigenvalues of

$$X = \begin{pmatrix} 1 & 1 & 1 & .... & 1 \\ 1 & 1 & 1 & .... & 1 \\ 1 & 1 & 1 & .... & 1 \\ .. & .. & .. & .... & .. \\ .. & .. & .. & .... & .. \\ 1 & 1 & 1 & .... & 1 \end{pmatrix}$$

(i.e., $X_{ij} = 1, \quad i = 1...n, \ j = 1...n.$).

What is the Jordan canonical form of this matrix?

**8+2**

Eigenvalues : soln of the eqn $\det |X - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 & \cdots & 1 \\ 1 & 1-\lambda & 1 & \cdots & 1 \\ \cdots & & & & \\ 1 & 1 & 1 & \cdots & 1-\lambda \end{vmatrix} = 0$$

adding columns $2 \cdots n$ to 1st column

$$\det = \begin{vmatrix} n-\lambda & 1 & 1 & \cdots & 1 \\ n-\lambda & 1-\lambda & 1 & \cdots & 1 \\ \vdots & & & & \\ n-\lambda & 1 & 1 & \cdots & 1-\lambda \end{vmatrix} = (n-\lambda) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1-\lambda & 1 & \cdots & 1 \\ 1 & 1 & 1-\lambda & \cdots & 1 \\ \cdots & & & & \\ 1 & 1 & 1 & \cdots & 1-\lambda \end{vmatrix}$$

Subtracting 1st column from columns $2 \cdots n$

$$\det = (n-\lambda) \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -\lambda & 0 & \cdots & 0 \\ 1 & 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 1 & 0 & 0 & \cdots & -\lambda \end{vmatrix} = (n-\lambda)(-\lambda)^{n-1}$$



$\Rightarrow$ eigenvalues of $X$ are $n$, and $0$ with $(n-1)$ fold degeneracy.

Now $X$ is real symmetric $\Rightarrow$ can be diagonalized by orthogonal transform

$$\Rightarrow \text{Jordan form} = \text{diagonal form} = \begin{pmatrix} n & & & \\ & 0 & & 0 \\ & & 0 & \\ & 0 & & \ddots \end{pmatrix}.$$

<u>Notes</u> Some other tricks: $X^2 = nX$ (simple matrix multiplication)

$\Rightarrow$ The eigenvalues should satisfy $\lambda^2 = n\lambda$ (as obtained by acting on suitable eigenvectors)

$$\Rightarrow \lambda = n, 0$$

Now trace = sum of eigenvalues $= n$

$\Rightarrow$ eigenvalues $= n$, and $0$ with $(n-1)$ fold degeneracy

Even simpler is just to realize that the eigenvectors are simple to get: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ is the eigenvector with eigenvalue $n$,

and $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ \vdots \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$ are $(n-1)$ L.I. eigenvectors with eigenvalue $0$

[one of you [redacted] did it this way]

Since you get all the eigenvectors, Jordan form = diagonal form.

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3. Consider matrices of the form

$$A = \begin{pmatrix} m & k \\ 0 & n \end{pmatrix}$$

where $m, n, k$ are integers mod 5, and integer operations (addition / multiplication) are defined mod 5.

a) Show that the set of all $A$ form a vector space.

**8**

b) Let G=set of all A such that $m, n \neq 0$. Show that G forms a group under matrix multiplication. What is the order of the group?

**8**

c) Is G simple/semisimple?

**2**

a) First we convince ourselves that $F = \{0,1,2,3,4\}$, with addition and multiplication defined mod 5, form a field:

$c_1 \oplus c_2 \bmod 5 \in F \qquad c_1 c_2 \bmod 5 \in F$

multiplication, addition satisfy commutivity and associativity

$0 \oplus c = c \quad \forall c$

$\exists -c$ :

| $c$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $-c$ | 0 | 4 | 3 | 2 | 1 |

$\rightarrow$ 1

$c \otimes 1 = c \quad \forall c$

for $c \neq 0$, $\exists c^{-1}$ :

| $c$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $c^{-1}$ | 1 | 3 | 2 | 4 |

$\rightarrow$ 2

$c \otimes (c_2 \oplus c_3) = c_1 c_2 \oplus c_1 c_3$

Now it is easy to show that $A = \begin{pmatrix} m & k \\ 0 & n \end{pmatrix}$ form vector space over this field F, when addition defined as

$$A_1 \oplus A_2 = \begin{pmatrix} m_1 \oplus m_2 & k_1 \oplus k_2 \\ 0 & n_1 \oplus n_2 \end{pmatrix}, \text{ is of same form}$$

Commutivity and associativity of vector addition follow immediately

$$\text{null vector} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad -A = \begin{pmatrix} -m & -k \\ 0 & -n \end{pmatrix} \text{ with } -m \text{ etc defined in (1)}$$

and properties of scalar multiplication follow immediately.

b) $G = \{A\}$ with $m, n \neq 0$

$$A_1 A_2 = \begin{pmatrix} m_1 & k_1 \\ 0 & n_1 \end{pmatrix} \begin{pmatrix} m_2 & k_2 \\ 0 & n_2 \end{pmatrix} = \begin{pmatrix} m_1 \otimes m_2 & m_1 \otimes k_2 \oplus k_1 \otimes n_2 \\ 0 & n_1 \otimes n_2 \end{pmatrix} \in G$$

identity : $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

inverse : $A^{-1} = \begin{pmatrix} m' & k' \\ l' & n' \end{pmatrix}$ such that $A^{-1} A = I = \begin{pmatrix} m'm & m'k + k'n \\ l'm & l'k + n'n \end{pmatrix}$

since $l'm = 0$, $l' = 0 \Rightarrow m' = m^{-1}$, $n' = n^{-1}$ (as defined in (2))

$m'k + k'n = 0 \Rightarrow k' = n^{-1} m' k \Rightarrow A^{-1} = \begin{pmatrix} m^{-1} & n^{-1} \otimes m^{-1} \otimes k \\ 0 & n^{-1} \end{pmatrix}$

Easy to check that $AA^{-1} = I$

$\Rightarrow$ inverse exists

Associativity follows trivially

$\Rightarrow$ G forms a group.

Order = # elements = 80 since m, n can take 4 values & k, 5 values.

c) $\mathcal{H} = \left\{ \begin{matrix} m & 0 \\ 0 & m \end{matrix} \right\}$, $m = 1, 2, 3, 4$ form an abelian invariant

subgroup of $G$ so $G$ is not simple / semisimple.

<u>Notes</u>

This was a verbose question, I have given marks to ~~everybody~~ people who more-or-less made the correct points. Wanted to stress a couple of points though.

i) to have the multiplicative inverses, it was important that 5 is a prime. If you had operations mod 6, for example, then ~~you~~ you ~~group~~ would not have a group (no inverse of $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, for example).

ii) Very few of you got c) correct. Please try to think it through.

(10)

4. Let $f(x,y)$ be a complex function of the real variables $x$ and $y$, and let $Ref = u$, $Imf = v$. Find the necessary relations (equivalent of Cauchy-Riemann conditions) between partial derivatives of $u$ and $v$ for $f$ to be a differentiable function of $z^* = x - iy$.

**10**

For $f(x,y) = u(x,y) + i\,v(x,y)$ to be a differentiable fn. of $z^* = x - iy$, derivatives $\frac{\partial}{\partial z^*}$ taken along different directions in x-y plane should agree.

Now varying $z^*$ such that y is fixed,

$$\frac{\partial f}{\partial z^*} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + i\,v(x+\Delta x, y) - u(x,y) - i\,v(x,y)}{\Delta z^* = \Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

varying $z^*$ such that x is fixed,

$$\frac{\partial f}{\partial z^*} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + i\,v(x, y+\Delta y) - u(x,y) - i\,v(x,y)}{\Delta z^* = -i\Delta y} = i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= -\frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

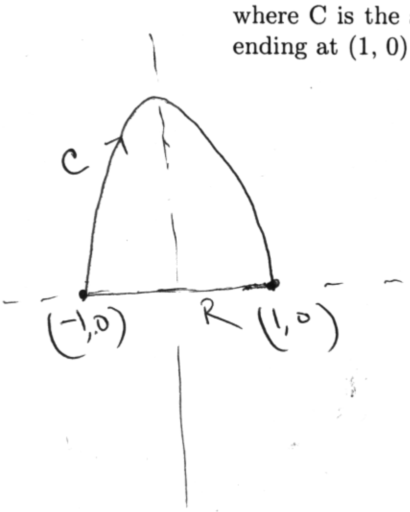
These are the equivalents of the CR conditions.

5. Find the value of the integral

$$\int_C \frac{z}{z^4-16}dz$$

where C is the semi-ellipse $9x^2+y^2=9$ in the upper half-plane ($y>0$), starting at (-1, 0) and ending at (1, 0).

**12**




Let us call $C'$, the closed contour $C+R$

where $R$ is the straight line from $(1,0)$

to $(-1,0)$

$$\Rightarrow \oint_{C'} dz \frac{z}{z^4-16} = -2\pi i \,.\text{ sum of residues enclosed by } C'$$

[- sign because $C'$ is traversed clockwise] (*)

$$z^4-16 = (z+2i)(z-2i)(z+2)(z-2)$$

$\Rightarrow$ only pole of $\frac{z}{z^4-16}$ within $C'$ is at $z=2i$, residue $= \frac{2i}{4i\cdot(2+2i)(-2+2i)}$

$$= -\frac{1}{16}.$$

$$\Rightarrow \oint_{C'} dz \frac{z}{z^4-16} = -2\pi i\cdot\left(-\frac{1}{16}\right) = \frac{\pi i}{8}$$

$$= \int_C dz \frac{z}{z^4-16} + \int_R \frac{z}{z^4-16}dz$$



and $\int_R \frac{z}{z^4-16}dz = -\int_{-1}^{1} dx \frac{x}{x^4-16} = 0$ (odd fn. on even interval)

$$\Rightarrow \int_C dz \, \frac{z}{z^4 - 16} = \frac{\pi i}{8}.$$

Notes

Many of you have missed the minus sign at (*), or, taken the contour anticlockwise but then didn't correct for the fact that you've done the integration on $-C$ rather than $C$.

(13)

6. a) Find the singularity structure of the function

$$f(z) = \frac{\log\ z}{z^{1/4}\ (1+z)}$$

in the complex plane. Find the residues at the poles.

**8**

b) Use the above result to evaluate the integral

$$\int_0^\infty dx\ \frac{\log\ x}{x^{1/4}\ (1+x)}.$$

**12**

a) $f(z)$ has branch points at $z=0$ and $\infty$, and a simple pole at $z=-1$. ~~Residues~~

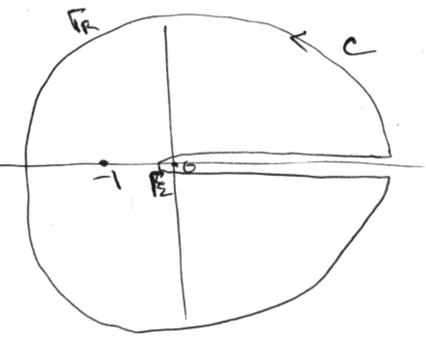
Taking the branch $0 \leq \theta < 2\pi$ (branch cut along +ve real axis),

pole is at $z = e^{i\pi}$ and residue $= \dfrac{i\pi}{e^{i\pi/4}}$

$$= \frac{i\pi}{(1+i)/\sqrt{2}} = \frac{i\pi(1-i)}{\sqrt{2}}$$

$$= \frac{\pi(1+i)}{\sqrt{2}}.$$

b) Let us calculate $I = \oint_C dz\ \dfrac{\log z}{z^{1/4}(1+z)}$, where $C$ is the contour shown.




Using Cauchy formula, $I = \dfrac{2\pi i \cdot \pi(1+i)}{\sqrt{2}} = \sqrt{2}\,\pi^2\,(i-1)$

$\rightarrow$ (1)

Now $I = \int_\varepsilon^R dx \, \frac{\ln x}{x^{1/4}(1+x)} + \int_{\Gamma_R} dz \, \frac{\ln z}{z^{1/4}(1+z)} + \int_R^\varepsilon dx \, \frac{\ln x + 2\pi i}{x^{1/4} e^{i\pi/2}(1+x)} + \int_{\Gamma_\varepsilon} dz \, \frac{\ln z}{z^{1/4}(1+z)}$

where $\Gamma_R$ : circle of radius $R$ traversed anticlockwise, and $\Gamma_\varepsilon$ : circle of radius $\varepsilon$ at origin, traversed clockwise.

$$\int_{\Gamma_R} dz \, \frac{\ln z}{z^{1/4}(1+z)} = \int_0^{2\pi} \frac{Re^{i\theta} \, i\,d\theta \,.\, (\ln R + i\theta)}{R^{1/4} e^{i\theta/4}(1+Re^{i\theta})}$$

$$\left| \int_0^{2\pi} \frac{Re^{i\theta} \, i\,d\theta \, \ln R}{R^{1/4} e^{i\theta/4}(1+Re^{i\theta})} \right| \leq 2\pi \,.\, \frac{R \ln R}{R^{1/4}|R-1|} \xrightarrow[R\to\infty]{} \frac{2\pi \ln R}{R^{1/4}} \longrightarrow 0$$

$$\left| \int_0^{2\pi} \frac{Re^{i\theta} \, i\,d\theta \,.\, i\theta}{R^{1/4} e^{i\theta/4}(1+Re^{i\theta})} \right| \leq 4\pi^2 \,.\, \frac{R}{R^{1/4}|R-1|} \xrightarrow[R\to\infty]{} 0$$

$$\int_{\Gamma_\varepsilon} dz \, \frac{\ln z}{z^{1/4}(1+z)} = -\int_0^{2\pi} \varepsilon e^{i\theta} \, i\,d\theta \,.\, \frac{\ln\varepsilon + i\theta}{\varepsilon^{1/4} e^{i\theta/4}(1+\varepsilon e^{i\theta})} \xrightarrow[\varepsilon\to 0]{} -\varepsilon^{3/4} \int_0^{2\pi} (\ln\varepsilon + i\theta) e^{i3\theta/4} \, i\,d\theta$$

$$\longrightarrow 0$$

$\Rightarrow$ In the limit $\varepsilon\to 0, R\to\infty$

$$I = \int_0^\infty dx \, \frac{\ln x}{x^{1/4}(1+x)} - \int_0^\infty \frac{dx\,(\ln x + 2\pi i)}{x^{1/4} \,.\, i \,.\, (1+x)}$$

$$= (1+i) \int_0^\infty \frac{dx \, \ln x}{x^{1/4}(1+x)} - \int_0^\infty \frac{dx \,.\, 2\pi}{x^{1/4}(1+x)}$$

$$= \sqrt{2}\,\pi^2 (i-1) \quad \text{using (1)} \longrightarrow (2)$$

Now evaluating $\int_C dz \, \frac{1}{z^{1/4}(1+z)}$ following the same steps, we get

$$\int_0^\infty \frac{dx}{x^{1/4}(1+x)} + i \int_0^\infty \frac{dx}{x^{1/4}(1+x)} = 2\pi i \,.\, \frac{\sqrt{2}}{1+i} = \pi(1+i)\sqrt{2} \Rightarrow \int_0^\infty \frac{dx}{x^{1/4}(1+x)} = \sqrt{2}\,\pi$$

Putting this in (2), we get

$$(1+i)\int_0^\infty \frac{dx\,\ln x}{x^{1/4}(1+x)} - 2\pi\cdot\sqrt{2}\pi = \sqrt{2}\pi^2(i-1)$$

$$\Rightarrow (1+i)\int_0^\infty \frac{dx\,\ln x}{x^{1/4}(1+x)} = \sqrt{2}\pi^2(i+1)$$

$$\Rightarrow \int_0^\infty \frac{dx\,\ln x}{x^{1/4}(1+x)} = \sqrt{2}\pi^2.$$

Notes i) This is how I had done it, but it is not the smartest way. At Eq. (2), you can match the imaginary part of both sides to get

$$\int_0^\infty \frac{dx\,\ln x}{x^{1/4}(1+x)} = \sqrt{2}\pi^2.$$

(Some of you did it this way — thanks!)

ii) Quite a few of you got to Eq. (2), but then just dropped $\int_0^\infty dx \frac{2\pi}{x^{1/4}(1+x)}$. Don't drop it just because it looks troublesome!

iii) A surprising no. of people got confused evaluating $(-1)^{1/4}$. Once you've chosen the branch to be $\theta = 0 - 2\pi$, $-1$ is $e^{i\pi}$ (no more multivaluedness), and $(-1)^{1/4}$ is uniquely defined: $e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$
$= \frac{1+i}{\sqrt{2}}$.

7. Let $u(x,y) = x - (x^2 - y^2)/2$. Find a real function $v(x,y)$ such that $f(z = x+iy) = u(x,y) + iv(x,y)$ is analytic, and $f(0) = 0$.

Find the saddle points of $f(z)$. Find the directions along the saddle points such that $u$ is nearly constant and $v$ has a local maximum (along the trajectory) at the saddle point.

**5+5+5**

$v$ has to satisfy C-R conditions:

$$\left.\frac{\partial v}{\partial y}\right|_x = \left.\frac{\partial u}{\partial x}\right|_y = 1 - x \implies v = y(1-x) + g(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -y \implies v = y(1-x) + c$$

$$f(0) = 0 \implies c = 0 \text{ and } f = x - \tfrac{1}{2}(x^2 - y^2) + iy(1-x)$$
$$= z - \tfrac{1}{2} z^2$$

saddle point: $\left.\frac{\partial f}{\partial z}\right|_{z_0} = 1 - z_0 = 0 \implies z_0 = 1$.

Expanding to quadratic order around saddle point,

$$f(z) = f(z_0) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial z^2}\right|_{z_0} (z - z_0)^2 + \ldots \qquad f(z_0) = \tfrac{1}{2},\ \left.\frac{\partial^2 f}{\partial z^2}\right|_{z_0} = -1$$

$$= \frac{1}{2} - \frac{1}{2}(z - z_0)^2 + \ldots$$

For the trajectory through $z_0$ such that $u$ nearly constant and $v$ has a maximum, we look for trajectories like $f(z_0) - i\frac{t^2}{2}$, $t$ real

$$\implies (z - z_0)^2 \sim it^2 \implies z - z_0 \sim t e^{i\pi/4}$$

$\implies$ the required trajectory is $1 + t e^{i\pi/4}$,

Notes i) Many of you immediately recognized $u$ to ~~R~~ be $\mathrm{re}\left(z - \frac{1}{2}z^2\right)$,

and bypassed C-R route. Cool!

ii) Very few people (only one?) got the trajectory correct.

Many of you instead found the trajectory where $u$ has a maximum

and $v$ is nearly constant. Please read the question!