Complex variables

Note : complex no. $z = x+iy = re^{i\theta}$

$$z^* = x-iy = re^{-i\theta}$$

$|z|^2 = x^2+y^2 = r^2 = zz^*$. Check $|z_1 z_2| = |z_1||z_2|$,

$$\big||z_1|-|z_2|\big| \le |z_1+z_2| \le |z_1|+|z_2|.$$

Can define limit, derivative in a way analogous to real calculus.

$w = \lim_{z\to z_0} f(z)$ if $\forall \varepsilon>0$, $\exists \delta>0$ such that if $|z-z_0|<\delta$ then $|f(z)-w|<\varepsilon$

If $\lim_{z\to z_0} f(z)$ exists and is equal to $f(z_0)$: f is continuous at z_0.

If $\lim_{\Delta z\to 0} \dfrac{f(z+\Delta z)-f(z)}{\Delta z}$ exists, f is called differentiable (holomorphic)

at z; the limit, called derivative, is denoted $\dfrac{df(z)}{dz}$ or $f'(z)$.

Properties (check) $(f+g)'(z) = f'(z)+g'(z)$.

$$(fg)'(z) = f'(z)g(z)+f(z)g'(z)$$

if f, g differentiable at z. Also if $g(z)\neq 0$,

$$\frac{d}{dz}\frac{f(z)}{g(z)} = \frac{f'(z)g(z)-f(z)g'(z)}{g(z)^2}.$$

using the above, eg., $\dfrac{d}{dz}z^n = nz^{n-1}$

If $w=f(z)$, f differentiable at z, and g differentiable at w,

then $\dfrac{d}{dz}g(f(z)) = g'(f(z))\,f'(z)$.

Proof : Since f differentiable at z, $f(z+\Delta z) - f(z) = f'(z)\Delta z + \sigma(z,\Delta z)\Delta z$

with $\sigma(z,\Delta z)\to 0$ as $\Delta z \to 0$.

Similarly, $g(\omega+k)-g(\omega)=g'(\omega)k+\rho(\omega,k)k$, with $\rho(\omega,k)\to 0$ as $k\to 0$

now putting $k=f'(z)\Delta z+\sigma(z,\Delta z)\Delta z$ we get

$$\lim_{\Delta z\to 0}\frac{g(f(z+\Delta z))-g(f(z))}{\Delta z}=g'(f(z))f'(z).$$

If $f(z)$ is differentiable everywhere on an open set $U\subset\mathbb{C}$,

we say f is differentiable (holomorphic) on U.

Say $f(z)=f(x+iy)=u(x,y)+iv(x,y)$. What is the condition of

differentiability in terms of u & v?

Need $\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$ to exist and be unique for different ways of

taking $\Delta z\to 0$ in (x,y) plane.

$$f'(z)=\lim_{\Delta x,\Delta y\to 0}\frac{u(x+\Delta x,y+\Delta y)+iv(x+\Delta x,y+\Delta y)-u(x,y)-iv(x,y)}{\Delta x+i\Delta y}$$

Taking $\Delta z\to 0$ along a line $\parallel$ to x axis (i.e., $\Delta y=0$, $\Delta x\to 0$)

$f'(z)=u_x+iv_x$

Now along $\Delta x=0$, $\Delta y\to 0$ $\Rightarrow f'(z)=-i(u_y+iv_y)$

$\Rightarrow$ necessary condition: $u_x=v_y$, $v_x=-u_y$ Cauchy-Riemann conditions.

Conversely, let $u(x,y)$ and $v(x,y)$ are two fns which are continuously differentiable in a neighborhood of (x_0,y_0) [i.e. the derivatives u_x,u_y,v_x,v_y exist and are continuous], and the derivatives satisfy C-R conditions.

Then $u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)=u'_x\Delta x+u'_y\Delta y+\sigma_1\Delta x+\sigma_2\Delta y$, $\sigma_1,\sigma_2\to 0$ as $\Delta x,\Delta y\to 0$

$v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0)=v'_x\Delta x+v'_y\Delta y+\rho_1\Delta x+\rho_2\Delta y$, $\rho_1,\rho_2\to 0$ as $\Delta x,\Delta y\to 0$

Forming $f(z) = u(x,y) + i\,v(x,y)$, Then,

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(u_x + i v_x)\Delta x + (u_y + i v_y)\Delta y + \sigma_1 \Delta x + \sigma_2 \Delta y + i\rho_1 \Delta x + i\rho_2 \Delta y}{\Delta x + i \Delta y}$$

$$= (u_x + i v_x) + \lim_{\Delta x, \Delta y \to 0} \frac{(\sigma_1 + i\rho_1)\Delta x + (\sigma_2 + i\rho_2)\Delta y}{\Delta x + i\Delta y}$$

(using C-R conditions)

or, $$\left| \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} - (u_x + i v_x) \right| = \lim_{\Delta x, \Delta y \to 0} \left| \frac{(\sigma_1 + i\rho_1)\Delta x + (\sigma_2 + i\rho_2)\Delta y}{\Delta x + i\Delta y} \right|$$

$$\leq \lim_{\Delta x, \Delta y \to 0} \left[|\sigma_1 + i\rho_1| \frac{|\Delta x|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} + |\sigma_2 + i\rho_2| \frac{|\Delta y|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right]$$

$$= 0.$$

$\Rightarrow f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists and is equal to $u_x + i v_x$.

Example Take $f(z) = x - iy \Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \Rightarrow$ not differentiable.

$f(z) = z^2 = (x^2 - y^2) + i \cdot 2xy \Rightarrow$ CR condn satisfied and $f'(z) = 2x + i2y = 2z$.

Example where CR condn satisfied but ~~f(z)~~ $u + iv$ not differentiable:

$v(x,y) = -x+y,\ u(x,y) = x+y$ if $x=0$ and/or $y=0$

$u, v = 1$ else

at $(0,0)$ CR conditions satisfied, but in no neighborhood of $(0,0)$ the partial derivatives exist. Also quite clearly $f(z)$ not differentiable at $z=0$.

Now write $u(x,y)+iv(x,y) = u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right) + iv\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right) = g(z,z^*)$

What do the CR conditions translate to in terms of derivatives wrto z, z^*?

$$\left.\frac{\partial g}{\partial z^*}\right|_z = \left.\frac{\partial x}{\partial z^*}\right|_z \left.\frac{\partial g}{\partial x}\right|_y + \left.\frac{\partial y}{\partial z^*}\right|_z \left.\frac{\partial g}{\partial y}\right|_x = \frac{1}{2}(u_x+iv_x) + \frac{i}{2}(u_y+iv_y) = 0$$

$$\left.\frac{\partial g}{\partial z}\right|_{z^*} = u_x + iv_x$$

$\Rightarrow$ to be holomorphic, f is fn of z only and not z^*.

CR conditions in polar form: writing $f(z) = u(x,y) + iv(x,y)$ ~~...~~

$= u(x(r,\theta), y(r,\theta)) + iv(x(r,\theta), y(r,\theta))$

$$\Rightarrow \left.\frac{\partial u}{\partial r}\right|_\theta = \left.\frac{\partial x}{\partial r}\right|_\theta \left.\frac{\partial u}{\partial x}\right|_y + \left.\frac{\partial y}{\partial r}\right|_\theta \left.\frac{\partial u}{\partial y}\right|_x = \frac{x}{r}u_x + \frac{y}{r}u_y$$

$$\frac{1}{r}\left.\frac{\partial u}{\partial \theta}\right|_r = \frac{1}{r}\left.\frac{\partial x}{\partial \theta}\right|_r \left.\frac{\partial u}{\partial x}\right|_y + \frac{1}{r}\left.\frac{\partial y}{\partial \theta}\right|_r \left.\frac{\partial u}{\partial y}\right|_x = -\frac{y}{r}u_x + \frac{x}{r}u_y$$

$$\left.\frac{\partial v}{\partial r}\right|_\theta = \frac{x}{r}v_x + \frac{y}{r}v_y, \quad \frac{1}{r}\left.\frac{\partial v}{\partial \theta}\right|_r = -\frac{y}{r}v_x + \frac{x}{r}v_y$$

$\Rightarrow$ (for $r \neq 0$) CR conditions $\Rightarrow \left.\frac{\partial u}{\partial r}\right|_\theta = \frac{1}{r}\left.\frac{\partial v}{\partial \theta}\right|_r, \quad \left.\frac{\partial v}{\partial r}\right|_\theta = -\frac{1}{r}\left.\frac{\partial u}{\partial \theta}\right|_r$.

Ex for $f(z) = z^2$, $u = x^2 - y^2 = r^2\cos 2\theta$, $v = r^2 \sin 2\theta \Rightarrow$ CR satisfied

for $f(z) = z^{*2}$, $v = -r^2 \sin 2\theta \Rightarrow$ CR not satisfied.

Complex series: Series like $\sum_{n=0}^{\infty} z_n$. Converges if partial sums converge to a limit. Formally, series converges to z if given ϵ, $\exists N$ such that $\left| z - \sum_{n=0}^{M} z_n \right| < \epsilon$ for $M \geq N$.

Clearly, for series to converge, $\lim_{n\to\infty} z_n = 0$.

If $\sum_{n=0}^{\infty} |z_n|$ converges, series $\sum_{n=0}^{\infty} z_n$ said to "converge absolutely"

$\Rightarrow$ Then can use usual tests of real series.

Comparison test: if $|z_n| \leq a_n$ and $\sum_n a_n$ converges, then $\sum_n z_n$ converges absolutely.

If $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = c$, if $c < 1$, series $\sum_n z_n$ converges absolutely.

If $\lim_{n\to\infty} \left(|z_n|\right)^{1/n} = c$, if $c < 1$, series converges absolutely.

Power series: Series like $\sum_n a_n z^n$.

We will also consider series of fns, $\sum_{n=0}^{\infty} f_n(z)$. We will say such a series converges (uniformly) for $z \in$ some domain U if given ϵ, $\exists N$ such that $\left| f(z) - \sum_{n=0}^{M} f(z) \right| < \epsilon$ for $M \geq N$.

Let $f(z)$ be defined in a region including z_0. f is analytic at z_0 if in a neighborhood of radius $r > 0$ (i.e. $\forall z$ such that $|z-z_0| < r$) f has a power series expansion: $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$.

If $f(z)$ is analytic at every point of an open set U, f is said to be analytic on U.

If the power series converges absolutely for $|z-z_0|<r$, and does not converge absolutely for $|z-z_0|>r$, r is called the radius of convergence of the series.

[If $\sum z_n$ converge, $\lim_{n\to\infty} z_n = 0$]

Can show $\frac{1}{r} = \lim_{n\to\infty} |a_n|^{1/n}$.

Proof can take $z_0 = 0 \Rightarrow \sum a_n z^n$

for some $s<r$, since $\sum |a_n| s^n$ converges, $|a_n| s^n \to 0$ as $n \to \infty$

$\Rightarrow \exists\, C$ such that $|a_n| s^n \leq C \quad \forall n$

Then $|a_n|^{1/n} \leq \frac{C^{1/n}}{s}$ so $\lim_{n\to\infty} |a_n|^{1/n} \leq \frac{1}{s}$.

Since s is any +ve no. $<r$, $\lim_{n\to\infty} |a_n|^{1/n} \leq \frac{1}{r}$.

If $|a_n|^{1/n} = t < \frac{1}{r}$, take s such that $t < \frac{1}{s} < \frac{1}{r}$

Then $|a_n|^{1/n} s < 1$ for all but a finite # of $n \Rightarrow \sum_n |a_n| s^n$ converges

which is contradictory since $s > r$. $\Rightarrow \lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{r}$.

Example take $\sum z^n \Rightarrow \lim |a_n|^{1/n} = 1 \Rightarrow r = 1$

$\sum n^n z^n \Rightarrow r = 0$

$\sum n^{-n} z^n \Rightarrow r = \infty$

Let $f(z) = \sum a_n (z-z_0)^n$ is a power series with radius of convergence r.

Then $f(z)$ is analytic on the open disc of radius r, centered around z_0.

Proof We need to show that f has a power series expansion at an arbitrary point ω in the open disc $D(z_0, r)$, i.e. if $s > 0$ such that $|\omega - z_0| + s < r$, f has a power series expansion at ω, converging absolutely on the disc of radius s.

Without loss of generality set $z_0 = 0$.

Write $z = \omega + (z - \omega) \Rightarrow z^n = (\omega + (z-\omega))^n = \sum_{k=0}^{n} {}^nC_k\, \omega^{n-k} (z-\omega)^k$

Since $\sum a_n z^n$ converges absolutely, $\sum_n |a_n| \sum_{k=0}^{n} {}^nC_k\, |\omega|^{n-k} |z-\omega|^k$ converges. $\Rightarrow \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} a_n\, {}^nC_k\, |\omega|^{n-k} \right] |z-\omega|^k$ converges

$\Rightarrow \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} a_n\, {}^nC_k\, |\omega|^{n-k} \right] (z-\omega)^k$ converges absolutely.

If $f(z) = \sum a_n z^n$ has radius of convergence r, the series $\sum n a_n z^{n-1}$ has same radius of convergence: $\lim_{n\to\infty} |n a_n|^{1/n} = \lim_{n\to\infty} n^{1/n} |a_n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n}$

Theorem If $f(z) = \sum_n a_n (z - z_0)^n$ has radius of convergence r, f is holomorphic on $D(z_0, r)$ and $f'(z) = \sum_n n\, a_n (z - z_0)^{n-1}$.

Proof Setting $z_0 = 0$, and taking z inside $D(0, r)$, let us choose δ such that $|z| + \delta < r$.

for $|\Delta z| < \delta$, $f(z + \Delta z) = \sum a_n (z + \Delta z)^n = \sum a_n z^n + \sum n a_n z^{n-1} \Delta z + (\Delta z)^2 \sum_n R_n(z, \Delta z)$

where $R_n(z, \Delta z) = \sum_{k=2}^{n} {}^nC_k (\Delta z)^{k-2} z^{n-k}$ and $|R_n(z, \Delta z)| \le R_n(|z|, \delta)$.

$$\Rightarrow \frac{f(z+\Delta z)-f(z)}{\Delta z} - \sum n a_n z^{n-1} = (\Delta z) \sum a_n P_n(z, \Delta z) \to 0 \text{ as } \Delta z \to 0$$

— o —

$\Rightarrow f'(z_0) = a_1$ similarly, $a_n = \frac{f^{n}(z_0)}{n!}$.

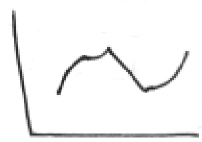
Very similarly, can show that $\sum_n \frac{a_n}{n+1}(z-z_0)^{n+1} = g(z)$ also converges and $g'(z) = f(z)$.

Complex integration

We first define a curve in the complex plane:

$$z(t) = x(t) + i y(t) \quad t_a \le t \le t_b$$

Let us assume derivatives are defined: $z'(t) = x'(t) + i y'(t)$ and they are continuous. A generalization:



path: a sequence of curves $\{z_1 \dots z_n\}$ such that

$$z_1(t_b^1) = z_2(t_a^2) \text{ etc.}$$

We can define integrals of $f(z) = u(t) + i v(t)$: $\int f(z(t))\,dt = \int u(t)\,dt + i\int v(t)\,dt$

On a path $\gamma(t)$, $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$

<u>Example</u> Say $f(z) = \frac{1}{z}$, $\gamma = e^{i\theta}$ (unit circle) $\Rightarrow \int_\gamma f(z)\,dz = \int_0^{2\pi} e^{-i\theta}\cdot i e^{i\theta}\,d\theta$
$= 2\pi i$.

Over a path, $\int_\gamma f(z)\,dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,dz$.

If f is continuous and $g' = f$, for any path γ connecting z_1, z_2

$$\int_\gamma f(z)\,dz = g(z_2) - g(z_1), \text{ since } \int f(z)\,dz = \int_1^2 g'(\gamma(t))\,\gamma'(t)\,dt$$
$$= \int_1^2 \frac{d}{dt} g(\gamma(t))\,dt.$$

This generalizes for a path.

In particular, if γ is closed, $\int_\gamma f(z)\,dz = 0$.

The length of a curve $\gamma(t)$: $L(\gamma) = \int_a^b |\gamma'(t)|\,dt$.

If f is continuous and $|f(\gamma(t))| \leq M$, then $\left|\int_\gamma f(z)\,dz\right| \leq ML$

Since $\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right| \leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt \leq ML$.

Generalization for non-smooth curve obvious.

<u>Cauchy's Theorem</u> For a ~~holomorphic~~ fn. $f(z)$ holomorphic on U and C a closed contour in U, $\oint f(z)\,dz = 0$.

The proof is trivial if we assume $f'(z)$ is continuous inside C:

$$\oint f(z)\,dz = \oint (u\,dx - v\,dy) + i\oint (v\,dx + u\,dy).$$

Now using Stokes's theorem, $\oint V_x\,dx + V_y\,dy = \int\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)dx\,dy$ and the C-R conditions, the result follows.

<u>Example</u> take γ: unit circle and $f(z) = z^2$

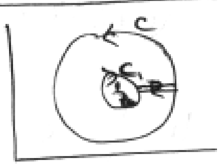
$$\Rightarrow \oint f(z)\,dz = \int_0^{2\pi} e^{2i\theta}\cdot e^{i\theta}\, i\,d\theta = 0.$$

Note: for $f(z) = \frac{1}{z}$, $\oint = 2\pi i$: $\frac{1}{z}$ not holomorphic inside the contour.

Generalize to a multiply connected domain: take the contour

$$\Rightarrow \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz = 0.$$

<u>Cauchy integral formula</u> If $f(z)$ is holomorphic on and in C, $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z - z_0}$



Surround z_0 by a circle C_1, centred at z_0 and radius ε (see left)

Then $\frac{f(z)}{z-z_0}$ holomorphic in the shown domain

$$\Rightarrow \oint_C \frac{f(z)\,dz}{z-z_0} + \oint_{C_1} \frac{f(z)\,dz}{z-z_0} + \int_{\uparrow\downarrow} \frac{f(z)\,dz}{z-z_0} = 0$$

$\frac{f(z)}{z-z_0}$ continuous along the connecting line $\Rightarrow \int_{\uparrow\downarrow} = 0 \Rightarrow \oint_C \frac{f(z)\,dz}{z-z_0} = -\oint_{C_1} \frac{f(z)\,dz}{z-z_0}$

$$\oint_{C_1} \frac{f(z)\,dz}{z-z_0} = \oint_{C_1} \frac{f(z_0+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\,\varepsilon e^{i\theta}\, i\,d\theta = i\int_{2\pi}^{0} f(z_0)\,d\theta \quad \text{as } \varepsilon \to 0$$

$$= 2\pi i\, f(z_0).$$

[formally, $\left|\oint_{C_1} \frac{f(z)\,dz}{z-z_0} - \oint_{C_1} \frac{f(z_0)\,dz}{z-z_0}\right| = \left|\oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right|$

$$\leq \frac{|f(z)-f(z_0)|_{max}}{\varepsilon}\, 2\pi\varepsilon$$

Since $f(z)$ is continuous around z_0, as $\varepsilon \to 0$ $|f(z)-f(z_0)|_{max} \to 0$

and so $\oint_{C_1} \frac{f(z)\,dz}{z-z_0} \to 2\pi i\, f(z_0)$.]

Now take Δz such that a disc of radius $|\Delta z|$ centred at z_0 is contained in C $\Rightarrow f(z_0+\Delta z) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z-z_0-\Delta z}$

$$\Rightarrow f(z_0+\Delta z) - f(z_0) = \frac{1}{2\pi i}\oint_C f(z)\,dz\left(\frac{1}{z-z_0-\Delta z} - \frac{1}{z-z_0}\right) = \frac{\Delta z}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0-\Delta z)(z-z_0)}$$

$$\Rightarrow f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^2}$$

Formally, $\lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} - \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^2} = \lim_{\Delta z \to 0} \frac{1}{2\pi i}\oint_C f(z)\,dz \cdot \frac{\Delta z}{(z-z_0)^2(z-z_0-\Delta z)}$

Say minimum distance of z_0 from C is d and maximum of $|f(z)|$ on C is M

then $\oint_C \frac{f(z)\,dz}{(z-z_0)^2(z-z_0-\Delta z)} \le \frac{M.L}{d^2(d-|\Delta z|)}$: finite as $\Delta z \to 0$

$$\Rightarrow f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \oint_c \frac{f(z)\,dz}{(z-z_0)^2}.$$

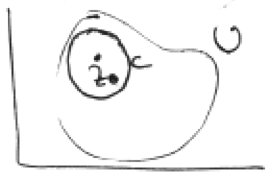
Similarly, $\lim_{\Delta z \to 0} \frac{f'(z_0+\Delta z) - f'(z_0)}{\Delta z} = \frac{1}{2\pi i} \lim_{\Delta z \to 0} \oint_c f(z)\,dz \cdot \frac{2(z-z_0) - (\Delta z)}{(z-z_0)^2 (z-z_0-\Delta z)^2}$

$$= \frac{2}{2\pi i} \oint_c \frac{f(z)\,dz}{(z-z_0)^3} = f''(z_0).$$

and $f^n(z_0) = \frac{n!}{2\pi i} \oint_c \frac{f(z)\,dz}{(z-z_0)^{n+1}}$.

$\Rightarrow$ derivatives of a ~~analytic~~ holomorphic fn. is also ~~analytic~~ holomorphic.

<u>Taylor expansion</u> Let $f(z)$ be holomorphic on an open ~~disc~~ set U, and let at a point z_0 in U, we ~~can~~ have a disc $C(z_0, R)$, centred at z_0 and radius R, contained in U. Then



$$f(z) = \frac{1}{2\pi i} \oint_c \frac{f(w)\,dw}{w-z} \quad \text{for } z \text{ in } C$$

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\left[1 + \frac{z-z_0}{w-z_0} + \cdots + \frac{(z-z_0)^n}{(w-z_0)^n} + \cdots\right]$$

$$\Rightarrow f(z) = f(z_0) + (z-z_0) f'(z_0) + \cdots + \frac{1}{n!}(z-z_0)^n f^n(z) + \cdots$$

$\Rightarrow$ if f holomorphic at z_0, it allows a power series expansion $\sum_n a_n (z-z_0)^n$.

$a_n = \frac{1}{2\pi i} \oint_c \frac{f(w)\,dw}{(w-z_0)^{n+1}} \le \frac{M}{R^n}$, where M is maximum of $f(z)$ on C

$\Rightarrow \lim_{n \to \infty} |a_n|^{1/n} \le \frac{1}{R} \Rightarrow$ radius of convergence $\geq R$.

[More carefully, $\frac{1}{w-z} = \frac{1}{w-z_0} \cdot \left[1 + \frac{z-z_0}{w-z_0} + \cdots + \left(\frac{z-z_0}{w-z_0}\right)^{n-1} + \frac{[(z-z_0)/(w-z_0)]^n}{1 - \frac{z-z_0}{w-z_0}}\right]$]

Contribution of last term $= (z-z_0)^n \oint \frac{f(w)\,dw}{(w-z_0)^n (w-z)} = R_n$

$\Rightarrow |R_n| = |(z-z_0)^n| \cdot \left|\oint_C \frac{f(\omega)\,d\omega}{(\omega-z_0)^n(\omega-z)}\right| \le \frac{M}{R-d}\cdot\left(\frac{d}{R}\right)^n$ where $d = |z-z_0|$

$\to 0$ as $n \to \infty$ since $d < R$.]

$\Rightarrow$ $\boxed{\text{analyticity} \Leftrightarrow \text{holomorphicity}}$

If the fn. is analytic over the whole complex plane, it is called entire fn..

Example: polynomial fns.

~~If~~ If an entire fn. is bounded, since $a_n \le \frac{M}{R^n}$, $a_n = 0$ for $n >$

$\Rightarrow$ a bounded entire fn. is constant (Liouville's theorem).

Now let $f(z)$ be a non-constant polynomial: $f(z) = c_0 + c_1 z + \cdots + c_n z^n$, $c_n \neq 0$. We can show that it always has a root. For, if it does not, then $g(z) = \frac{1}{f(z)}$ is entire. Also for large $|z|$, $g(z)$ is small $\Rightarrow$ bounded $\Rightarrow$ constant $\Rightarrow$ contradiction

$\Rightarrow$ Can iterate (expand around the zero, z_0) to show that $f(z)$ has n roots.

Goursat's proof of Cauchy's theorem

① If R is a rectangle, and f holomorphic on R, then $\int_{\partial R} f\,dz = 0$

divide R into 4 segments $\Rightarrow \int_{\partial R} f\,dz = \sum_{i=1}^{4} \int_{\partial R_i} f\,dz$

$$\Rightarrow \left|\int_{\partial R} f\,dz\right| \le \sum_{i=1}^{4} \left|\int_{\partial R_i} f\,dz\right|$$

$$\Rightarrow \exists i, \text{ say } 1, \text{ such that } \left|\int_{\partial R_1} f\,dz\right| \ge \frac{1}{4}\left|\int_{\partial R} f\,dz\right|$$

Now dividing R_1, and continuing, we get after n steps,

$$\left|\int_{\partial R^{(n)}} f\,dz\right| \ge \frac{1}{4^n}\left|\int_{\partial R} f\,dz\right|$$

Now continuing the process, we get $R^{(n)} \to z_0$ as $n \to \infty$.

Since f differentiable at z_0, we can have a disc, V, centred at z_0 such that $\forall z$ in the disc, $f(z) = f(z_0) + f'(z)(z-z_0) + (z-z_0)h(z)$, with $h(z) \to 0$ as $z \to z_0$

Take n large enough that $R^n \subset V \Rightarrow \int_{\partial R^{(n)}} f(z)\,dz = \int_{\partial R^{(n)}} f(z_0)\,dz + \int_{\partial R^{(n)}} f'(z)(z-z_0)\,dz$

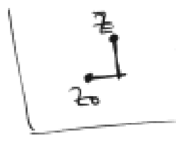
$$+ \int_{\partial R^{(n)}} (z-z_0)h(z)\,dz$$

$$= \int_{\partial R^{(n)}} (z-z_0)\,h(z)\,dz$$

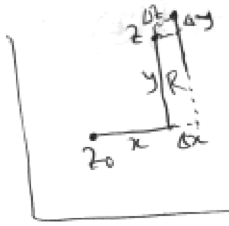
$$\Rightarrow \frac{1}{4^n}\left|\int_{\partial R} f(z)\,dz\right| \leq \left|\int_{\partial R^{(n)}} (z-z_0)\,h(z)\,dz\right| \leq \frac{2(a+b)}{2^n}\cdot\frac{a}{2^n}\cdot \max_{\partial R^{(n)}} h(z)$$

$$\Rightarrow \left|\int_{\partial R} f(z)\,dz\right| \leq \text{const.} \max_{\partial R^{(n)}} h(z) \to 0 \text{ as } n \to \infty.$$

(2) Now take a disc U centred around z_0. ~~such that~~ If f holomorphic on U, $\int_{\partial R} f = 0$ when $R \subset U$. Construct $g(z) = \int_{z_0,\lrcorner}^{z} f(w)\,dw$ where the integr. path is as shown:



We can show that $g(z)$ is differentiable, and that $g'(z) = f(z)$



$$g(z+\Delta z) - g(z) = \int_{z_0,\lrcorner}^{z+\Delta z} f(w)\,dw - \int_{z_0,\lrcorner}^{z} f(w)\,dw$$

$$= \int_{z_0}^{z_0+(x+\Delta x,0)} f(w)\,dw + \int_{z_0+(x+\Delta x,0)}^{z_0+(x+\Delta x,y+\Delta y)} f(w)\,dw - \int_{z_0}^{z_0+(x,0)} f(w)\,dw - \int_{z_0+(x,0)}^{z_0+(x,y)} f(w)\,dw$$

$$= \int_{z}^{z+(\Delta x,0)} f(w)\,dw + \int_{z+(\Delta x,0)}^{z+(\Delta x,\Delta y)} f(w)\,dw = \int_{z,\lrcorner}^{z+\Delta z} f(w)\,dw \text{ using } \int_{\partial R} f(w)\,dw = 0$$

Since $f(w)$ continuous, we can write $f(w) = f(z) + h(w)$, $\lim_{w\to z} h(w) = 0$

$$\Rightarrow \quad g(z+\Delta z) - g(z) = \Delta z\, f(z) + \int_{z,\perp}^{z+\Delta z} h(w)\, dw$$

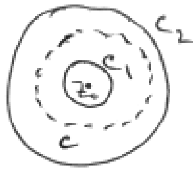
$$\Rightarrow \left| \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z} - f(z) \right| = \lim_{\Delta z \to 0} \left| \int_{z,\perp}^{z+\Delta z} h(w)\, dw \right| \le \lim_{\Delta z \to 0} \frac{|\Delta x| + |\Delta y|}{\Delta z} \max_{\perp} |h(w)| \to 0$$

$\Rightarrow$ $g(z)$ is an integral of f on U $\Rightarrow \displaystyle\int_{z_a}^{z_b} f(z)\, dz = g(z_b) - g(z_a)$

$$\Rightarrow \oint_C f(z)\, dz = 0.$$

—o—

<u>Laurent Series</u> Let $f(z)$ be analytic in the closed annular region betn. two discs C_1, C_2, as shown. Then following Cauchy's theorem,



$$f(z) = \frac{1}{2\pi i} \int_{C_2 - C_1} \frac{f(w)\, dw}{w - z}$$

$$= \frac{1}{2\pi i} \int_{C_2} \frac{f(w)\, dw}{w - z} - \frac{1}{2\pi i} \int_{C_1} \frac{f(w)\, dw}{w - z}$$

putting in 1st integral $\displaystyle \frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\left(1 + \frac{z-z_0}{w-z_0} + \frac{(z-z_0)^2}{(w-z_0)^2} + \cdots\right)$

and in the second, $\displaystyle -\frac{1}{w-z} = \frac{1}{(z-z_0)-(w-z_0)} = \frac{1}{z-z_0}\left(1 + \frac{w-z_0}{z-z_0} + \frac{(w-z_0)^2}{(z-z_0)^2} + \cdots\right)$

$$\Rightarrow f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{C_2} \frac{f(w)\, dw}{(w-z_0)^{n+1}} (z-z_0)^n + \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{C_1} \frac{f(w)\, dw}{(z-z_0)^n} (w-z_0)^{n-1}$$

$$= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

$$a_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)\, dw}{(w-z_0)^{n+1}} \quad \text{for } n \ge 0$$

$$= \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)\, dw}{(w-z_0)^{n+1}} \quad \text{for } n < 0$$

or, choosing contour C as shown,

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(\omega)\,d\omega}{(\omega - z_0)^{n+1}}, \qquad n = 0, \pm 1, \pm 2, \dots$$

Conversely, lets say f(z), analytic in the region, is given by the power series $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

Then $\oint f(z)\,dz\,.\,(z-z_0)^{-m-1} = \sum_{n=-\infty}^{\infty} a_n \oint dz\,(z-z_0)^{n-m-1}$

taking $C: z = z_0 + re^{i\theta}$, $\oint dz\,(z-z_0)^{n-m-1} = \int_0^{2\pi} re^{i\theta}\, i\,d\theta\,.\,r^{n-m-1} e^{i(n-m-1)\theta}$

$= 2\pi i\,\delta_{nm}$

$\Rightarrow a_m = \frac{1}{2\pi i}\oint \frac{f(z)\,dz}{(z-z_0)^{m+1}}$.

Laurent series Ex: $\frac{1}{z(z-1)} = -\frac{1}{z} - 1 - z - z^2 - \dots \quad 0<|z|<1, \quad \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots \quad |z|>1$.

Laurent series clearly shows nature of singularity at z_0.

Isolated singularity: if f(z) is analytic at every point in a neighborhood of z_0 except at z_0, then it is said to have an isolated singularity at z_0.

Now expanding $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

① if all $b_n = 0$, then fn (can be redefined to be) regular at z_0 [ex: $\frac{\sin z}{z}$]
[If f(z) bounded in a neighborhood of z_0]

② if $\exists$ m such that $b_m \neq 0$ and $b_n = 0\ \forall n > m$, Pole of order m at z_0

Ex: $\frac{1-z}{z^2}$, $\frac{e^z}{z}$

③ if $b_n \neq 0\ \forall n$, essential singularity Ex: $e^{1/z}$, $\sin(1/z)$

Example of non-isolated essential singularity: $\frac{1}{\sin(1/z)}$

$f(z)$ has pole of order m at z_0 iff $f(z)(z-z_0)^m$ analytic at z_0 and non-zero

If $f(z)$ is analytic on U ~~except~~ isolated poles, $f(z)$ is meromorphic on U.

If $f(z)$ has an isolated essential singularity at z_0, then $f(z)$ comes arbitrarily close to any complex no. in a neighborhood of z_0.

Let us translate z_0 to 0. $\Rightarrow$ in a neighborhood $0 < z < R$, called U, in which $f(z)$ is analytic, it is dense in $\mathbb{C}$. For, if not, say $\exists\, \alpha$ such that $|f(z) - \alpha| > s$ $\forall z \in U$, where s is a +ve no.

$\Rightarrow$ $g(z) = \dfrac{1}{f(z) - \alpha}$ is analytic and bounded on U $\Rightarrow$ can be extended to an analytic fn. on $\{0\} \cup U$ $\Rightarrow$ $\dfrac{1}{g(z)}$ has at most a pole at 0

$\Rightarrow$ $f(z)$ has at most a pole at 0.

<u>Residue</u> b_1 is called the residue of $f(z)$ at z_0. From its defn., when C has no other poles inside, $\oint_C f(z)\, dz = 2\pi i\, b_1$.

<u>Residue Theorem</u> If $f(z)$ is a (single valued) fn. analytic inside and on C except poles at z_i, with residues R_i, then $\oint_C f(z)\, dz = \sum_i 2\pi i\, R_i$

Proof: straightforward.

$\oint_{C - \Sigma C_i} f(z)\, dz = 0 \Rightarrow \oint_C f(z)\, dz = \sum_i \oint_{C_i} f(z)\, dz$.

Calculation of residues: If $f(z)=\frac{\varphi(z)}{(z-z_0)^m}$ near $z=z_0$, with $\varphi(z)$ analytic and nonzero at z_0, $\quad \text{res}\,(f(z))_{z_0} = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$.

More compactly, if $f(z)$ has pole of order m at z_0,

$$\text{res}\,(f(z))_{z_0} = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

In particular, if $f(z)=\frac{p(z)}{q(z)}$ has simple pole at z_0, residue $= \frac{p(z_0)}{q'(z_0)}$.

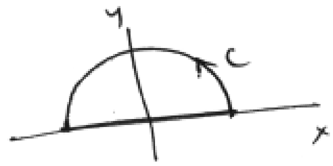
<u>Examples</u>

$$\frac{1}{(z+1)(z+3)^2} : \quad \text{res}\,(1) = \frac{1}{8}, \quad \text{res}\,(3) = -\frac{1}{8}$$

$$\tan z : \quad \text{res}\left(\left(n+\tfrac{1}{2}\right)\pi\right) = -1$$

Evaluation of real integrals:

Example: ① $\displaystyle\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$



$$\oint_C \frac{dz}{z^2+1} = \pi = \int_{-\infty}^{\infty}\frac{dx}{x^2+1} + \underbrace{\int_0^\pi \frac{Re^{i\theta}\, i\,d\theta}{R^2e^{2i\theta}+1}}_{I_R \,\leq\, \pi\cdot\frac{R}{R^2-1}\ \xrightarrow[R\to\infty]{}\ 0}$$

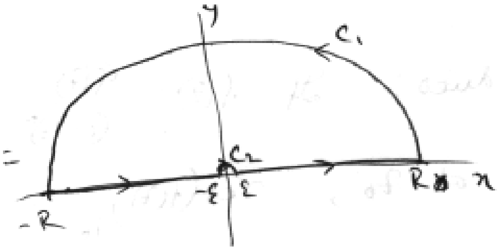
② $\displaystyle\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}$, $|b|<|a|$

on unit circle, $\sin\theta = \frac{z-\frac{1}{z}}{2i} \Rightarrow I = \oint \frac{dz/iz}{a+b\cdot\frac{z-1/z}{2i}} = \frac{2}{b}\oint\frac{dz}{z^2+\frac{2ia}{b}z-1}$

Poles: $\underbrace{-\frac{ia}{b}\left(1+\sqrt{1-b^2/a^2}\right)}_{\text{outside } C}$, $-\frac{ia}{b}\left(1-\sqrt{1-b^2/a^2}\right)$

$$\Rightarrow I = 2\pi i\cdot\frac{2}{b}\cdot\frac{1}{\frac{2i}{b}\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

③ $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx$

Evaluate $\oint_C \frac{e^{iz}}{z}\,dz$, $C =$ 

$= 0$

$= \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx + \int_{C_1} \frac{e^{iz}}{z}\,dz + \int_{C_2} \frac{e^{iz}}{z}\,dz$

$\int_{C_2} \frac{e^{iz}}{z}\,dz = \int_{\pi}^{0} \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}}\,\varepsilon e^{i\theta}\, i\,d\theta \xrightarrow[\varepsilon \to 0]{} -\pi i$

$= \left[\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right] \frac{e^{ix}}{x}\,dx + \int_{C_1} \frac{e^{iz}}{z}\,dz + \int_{C_2}$

using Jordan's lemma, $\int_{C_1} \frac{e^{iz}}{z}\,dz = 0$

$\Rightarrow \lim_{\varepsilon \to 0} \left[\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx = \pi i \right]$

$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$.

Jordan's lemma if $|f(z)| \to 0$ uniformly as $|z| \to \infty$, $\int_{C_R} f(z)\, e^{iaz}\,dz \xrightarrow[R\to\infty]{} 0$

where C_R is semicircle of radius R in upper half plane and $a > 0$

[if $a < 0$, use lower half plane].

Proof : $I_R = \int_{C_R} f(Re^{i\theta})\, e^{iaR\cos\theta - aR\sin\theta}\, Re^{i\theta}\, i\,d\theta$

say $|f(Re^{i\theta})| < \varepsilon(R)$. $|I_R| \leq \varepsilon(R) \cdot \int_0^{\pi} d\theta\, Re^{-aR\sin\theta}$

[$\varepsilon(R) \to 0$ as $R \to \infty$].

$= 2\varepsilon(R) \cdot R \int_0^{\pi/2} d\theta\, e^{-aR\sin\theta}$

for $0 \leq \theta \leq \pi/2$, $\sin\theta \geq \frac{2\theta}{\pi}$ so $|I_R| \leq 2\varepsilon(R) R \int_0^{\pi/2} d\theta\, e^{-aR\cdot\frac{2\theta}{\pi}}$

$= 2R\varepsilon(R) \cdot \frac{1 - e^{-aR}}{2aR/\pi}$

$= \frac{\pi}{a}\varepsilon(R)\left(1 - e^{-aR}\right)$

$\to 0$ as $R \to \infty$.

The lemma is useful for calculating Fourier transforms of meromorphic fcn.

For $a=0$, $I_R \to 0$ only if $\lim_{R\to\infty} R\,\varepsilon(R) \to 0$.

Example Yukawa potential: Scattering of pions $|\cdots| \sim \frac{1}{\vec p^2+m^2}$

$$U(\vec r) \sim \int \frac{d^3p}{(2\pi)^3} \frac{e^{+i\vec p\cdot\vec r}}{\vec p^2+m^2}$$

$$= \frac{1}{4\pi^2}\int \vec p^2 dp\cdot d\cos\theta\cdot \frac{e^{+ipr\cos\theta}}{\vec p^2+m^2} = \frac{1}{4\pi^2}\cdot\int \frac{\vec p^2 dp}{\vec p^2+m^2}\cdot\frac{e^{+ipr}-e^{-ipr}}{ipr}$$

1st integral: $\frac{1}{4\pi^2 ir}\cdot\int \frac{dp\ p}{\vec p^2+m^2}\, e^{+ipr} = \frac{1}{4\pi^2 ir}(+2\pi i)\cdot\frac{+im}{+2im}\cdot e^{-mr}$

$$= \frac{1}{4\pi r}e^{-mr}$$

Same from 2nd term.

Cauchy principal value of integral: Say we calculate $\int_a^b \frac{f(x)}{x-x_0}dx$ as

$\lim_{\varepsilon\to 0}\left[\int_a^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^{b}\right]\frac{f(x)\,dx}{x-x_0}$, when $f(x)$ is analytic at x_0

If the limit exists, it is called Cauchy principal value:

$$P\int_a^b \frac{f(x)\,dx}{x-x_0} = \lim_{\varepsilon\to 0}\left[\int_a^{x_0-\varepsilon}+\int_{x_0+\varepsilon}^{b}\right]\frac{f(x)}{x-x_0}\,dx$$

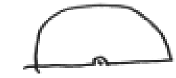
In the complex plane, we can indent the

contour: 

$$\int\frac{f(z)}{z-x_0}dz = P\int\frac{f(x)\,dx}{x-x_0} + \int_\pi^0 \frac{f(x_0+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\varepsilon e^{i\theta} i\,d\theta$$

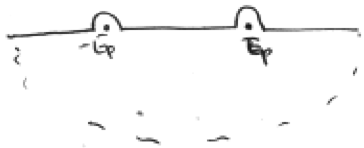
$$\to P\int\frac{f(x)\,dx}{x-x_0} - i\pi f(x_0)$$

If we indented  : $\int\frac{f(z)}{z-x_0}dz = P\int\frac{f(x)\,dx}{x-x_0} + i\pi f(x_0)$

eg. $\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx$:  : $0 = P\int\frac{e^{ix}}{x}dx - i\pi$

 : $2\pi i = P\int\frac{e^{ix}}{x}dx + i\pi$

Consider, e.g., $\int dp_0 \dfrac{e^{-ip_0 t}}{p_0^2 - E_p^2}$



: if $t>0$: $(-2\pi i)\left[\dfrac{e^{-iE_p t}}{2E_p} - \dfrac{e^{iE_p t}}{2E_p}\right]$

$$= -2\pi \frac{\sin E_p t}{E_p}$$

if $t<0$: 0

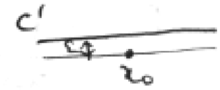
: if $t>0$: 0 if $t<0$: $2\pi \sin \dfrac{E_p t}{E_p}$

: if $t>0$: $-2\pi i \cdot \dfrac{e^{-iE_p t}}{2E_p}$

$t<0$: $-2\pi i \dfrac{e^{iE_p t}}{2E_p}$

Note that doing the integral

$$I = \int_C \frac{f(z)}{z-x_0}\,dz \equiv \int_{C'} \frac{f(z)\,dz}{z-x_0}$$



$$= \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{f(z)\,dz}{z-x_0}$$

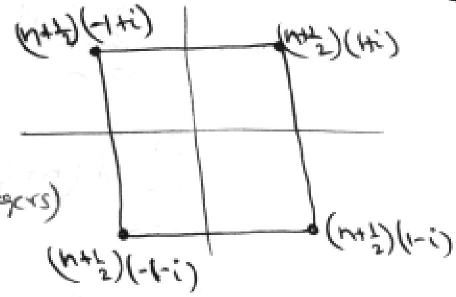
Substituting $z' = z - i\varepsilon$: $I = \displaystyle\int_{-\infty}^{\infty} \frac{f(z'+i\varepsilon)\,dz'}{z'-x_0+i\varepsilon} = \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-x_0+i\varepsilon}$

$$\Rightarrow \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-x_0+i\varepsilon} = P\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x-x_0} - i\pi f(x_0)$$

Similarly $\displaystyle\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-x_0-i\varepsilon} = P\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x-x_0} + i\pi f(x_0)$

Sometimes written as $\displaystyle\lim_{\varepsilon\to 0} \frac{1}{x-x_0\pm i\varepsilon} = P\frac{1}{x-x_0} \mp i\pi\,\delta(x-x_0)$.

Evaluate $\oint_C \pi \cot \pi z \, f(z) \, dz$ on the contour



where $f(z)$ is analytic except poles at $a_1 \cdots a_k$ (not integers)

and $|z||f(z)| \to 0$ as $|z| \to \infty$

Then $\oint_C \to 0$ as $n \to \infty$: $|\cot \pi z| = \left| \dfrac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right|$ is bounded on C

Take, eg., $y > \frac{1}{2}$: $|\cot \pi z| = \left| \dfrac{e^{2i\pi x - 2\pi y} + 1}{e^{2i\pi x - 2\pi y} - 1} \right| \leq \dfrac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}}$

similarly for $y < \frac{1}{2}$ sector. It is also bounded in the $-\frac{1}{2} \leq y \leq \frac{1}{2}$ sector

Now $\cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$: simple poles at $z = n$

residues: $\lim_{z \to n\pi} \cos n\pi \cdot \dfrac{(z - n\pi)}{\sin z\pi} =$

$\Rightarrow \oint_C = 0 = \sum_{n=-\infty}^{\infty} f(n) + \sum$ residue at poles of $f(z)$

Example: $f(z) = \dfrac{1}{(a+z)^2} \Rightarrow 0 = \sum_{n=-\infty}^{\infty} \dfrac{1}{(a+n)^2} - \pi^2 \operatorname{cosec}^2 \pi a \Rightarrow \sum_{n=-\infty}^{\infty} \dfrac{1}{(a+n)^2} = \pi^2 \operatorname{cosec}^2 \pi a$

(residue at $z = -a$: $\dfrac{d}{dz}(\pi \cot \pi z)\big|_{z=-a} = -\pi^2 \operatorname{cosec}^2 \pi a$)

similarly, to evaluate $T \sum_{n=-\infty}^{\infty} \dfrac{1}{(2\pi n T)^2 + E^2}$

we take $f(z) = \dfrac{T}{E^2 + (2\pi T z)^2}$

poles $\Rightarrow z = \pm \dfrac{iE}{2\pi T}$ residue: $\pm \dfrac{T}{2 \cdot 2\pi T iE} \pi \cot \pm \dfrac{\pi i E}{2\pi T} = \dfrac{1}{4E} \dfrac{e^{-E/2T} + e^{E/2T}}{e^{-E/2T} - e^{E/2T}}$

$\Rightarrow T \sum_{n=-\infty}^{\infty} \dfrac{1}{(2\pi n T)^2 + E^2} = \dfrac{1}{2E} \left(\dfrac{e^{E/2T} + e^{-E/2T}}{e^{E/2T} - e^{-E/2T}} \right) = \dfrac{1}{2E} \left(1 + \dfrac{2}{e^{E/T} - 1} \right)$.

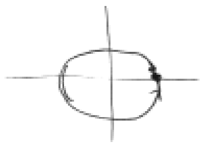
Similarly, $\sum_{n=-\infty}^{\infty}(-1)^n f(n) = -\sum$ residues of $\pi \operatorname{cosec}\pi z \cdot f(z)$ at poles of $f(z)$.

$$\sum_{n=-\infty}^{\infty} f(n+\tfrac{1}{2}) = \sum \text{residues of } \pi \tan \pi z \cdot f(z) \text{ at poles of } f(z)$$

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n+\tfrac{1}{2}) = \sum \text{residues of } \pi \sec \pi z\, f(z) \text{ at poles of } f(z)$$

$\int_0^\infty dx \frac{\sqrt{x}}{x^2+1}$ analyticity of $f(z) = \frac{\sqrt{z}}{z^2+1}$?

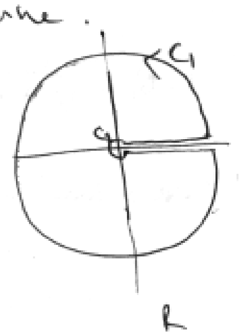
$\underset{=\sqrt{r}e^{i\theta/2}}{\sqrt{z}}$ is multivalued: taking a contour around $z=0$, we encounter a multivaluedness/discontinuity.



$\lim_{z\to x_0+i\varepsilon} \sqrt{z} = \sqrt{x_0}$, $\lim_{z\to x_0-i\varepsilon} \sqrt{z} = -\sqrt{x_0}$.

$z=0$ is called a branch point of $\sqrt{z}$. Can define the real axis as branch cut: discontinuity on crossing the line.

Define a contour not including this line:



$$\oint \frac{\sqrt{z}}{z^2+1}\,dz = 2\pi i \cdot \left[\frac{e^{i\pi/4}}{2i} + \frac{e^{3\pi i/4}}{-2i} \right] \qquad e^{i\pi/4} = \tfrac{1}{\sqrt{2}}(1+i),\quad e^{3\pi i/4} = \tfrac{1}{\sqrt{2}}(1-i)$$

$$= \frac{2\pi}{\sqrt{2}} = \underbrace{\int_{C_1} \frac{\sqrt{z}}{z^2+1}\,dz}_{\Downarrow\, 0} + \underbrace{\int_{C_2} \frac{\sqrt{z}}{z^2+1}\,dz}_{\Downarrow\, 0} + \int_\varepsilon^R \frac{\sqrt{x}}{x^2+1}\,dx + \int_R^\varepsilon \frac{-\sqrt{x}}{x^2+1}\,dx$$

$$\Rightarrow \int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx = \frac{\pi}{\sqrt{2}}.$$

① Choice of real axis for branch cut arbitrary: can choose any line joining $z=0$ to $z\to\infty$

② Makes sense to think of ∞ as one point, defined as mapping of 0 by the one-to-one fn. $\frac{1}{z}$: if $g(z) = \frac{1}{z}$ $(z \neq 0, \infty)$, properties of $f(z)$ at ∞ given by those of $g(z)$ at 0.

$g(z) = \frac{1}{\sqrt{z}}$: $z = 0$ is a branch point $\Rightarrow$ $z = \infty$ branch point of $f(z)$

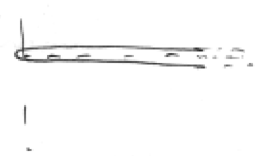
~~Cons~~ any line joining these branch points can be treated as branch cut: if we omit this ~~~~ line from defn. of the fn., no problem with multivaluedness | closed contour.

~~$\sqrt{z^2-1}$ $z = \pm 1$ branch points: branch cut joins those two points~~

~~A construction that allows to define arbitrary curves in the complex plane~~:

Say we take branch cut: +ve real axis

$\sqrt{z} = r$, $\theta \to 0^+ + \varepsilon$ $\quad$ $\sqrt{z} = -r$, $\theta = 2\pi - \varepsilon$



$\sqrt{z}$: upper half of complex plane

no closed curve crossing +ve real line (though such a curve encloses no singularity)

if we took branch cut : -ve real axis, range of $\sqrt{z}$ would be different

A construction that allows complex analysis everywhere in the plane for ~~~~ multivalued fn. : Riemann surface : define the two fns ~~~~

~~~~ $\sqrt{z}_{(1)} = re^{i\theta/2}$, $\theta = (-\pi, \pi)$

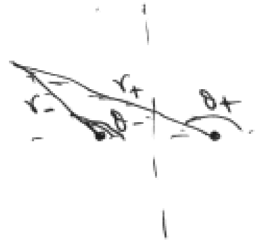
$\sqrt{z}_{(2)} = re^{i(\pi+\theta/2)}$, $\theta = (-\pi, \pi)$

So $\sqrt{z}_{(1)}\Big|_{\theta \to \pi - \varepsilon} = \sqrt{z}_{(2)}\Big|_{\theta \to -\pi + \varepsilon}$ etc. Now think of $\sqrt{z}_{(1)}$, $\sqrt{z}_{(2)}$ as $\sqrt{z}$ defined on ~~a~~ two sheets of the complex plane, joined crosswise along -ve real axis.

Could also ~~~~ join the sheets, e.g., along +ve real axis.
~~~~

$\sqrt{z^2-1}$: branch points at ± 1, branch cut : line joining the two

writing $\sqrt{z-1} = r_+ e^{i\theta_+/2}$, $\sqrt{z+1} = r_- e^{i\theta_-/2}$: $\sqrt{z^2-1} = \sqrt{r_+ r_-}\, e^{i(\theta_+ + \theta_-)/2}$



easy to check closed curves crossing the line joining ± 1 : multivaluedness problem

Riemann surfaces: two sheets with cuts along this line joined crosswise

$z^{1/n}$: branch points at $0, \infty$

Riemann surface : n sheets, i th sheet joined to $i \pm 1$, and n th sheet joined to $n-1$ th and 1st.

$\ln z$: $z = re^{i\theta}$, $\ln z = \ln r + i(\theta + \frac{2\pi k}{n})$ branch points: $0, \infty$

an infinite no. of sheets, joined along a branch cut

z^α : defined via $e^{\alpha \ln z}$: infinite sheets in Riemann surface, unless $\alpha = \frac{m}{n}$, m, n relative primes : n sheets.

Zeroes of an analytic fn : if f is analytic on U, and is non-zero, its zeroes are discrete.

Say, z_0 is a zero. We need to show that $\exists\, s>0$ such that $f(z) \neq 0$ in the disc $0<|z-z_0|<s$.

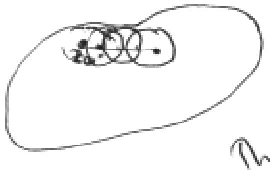
Since z_0 is in U and f analytic on U, we can expand in a power series:

$f = a_m(z-z_0)^m$ + higher orders, $m>0$, and this expansion has a finite radius of convergence

$\Rightarrow f = a_m(z-z_0)^m\left[1 + h(z-z_0)\right]$ where $h(z) = b_1 z + b_2 z^2 + \cdots$ has a finite radius of convergence

$\Rightarrow$ for small enough $|z-z_0|$, $|h(z-z_0)| < 1$ and $1 + h(z-z_0) \neq 0$.

$\Rightarrow$ If z_0 is not a discrete zero, the fn. vanishes in a neighborhood of z_0 $\Rightarrow$ vanishes within the radius of convergence of the Taylor series



$\Rightarrow$ vanishes everywhere in U (~~or~~ for any $z \in U$, we can find suitable points $z', z'', \dots$ and continue this argument till we reach z).

$\Rightarrow$ if two fns $f(z)$ and $g(z)$ are analytic on U, and are equal on a set of points S which are not discrete, then $f = g$

$\Rightarrow$ Let f is analytic on U and g on V, and U & V have a nonzero intersection and $f = g$ on $U \cap V$, then we call g the (unique) analytic continuation of U.

<u>Example:</u>

$$f_1(z) = 1 + z + z^2 + \dots = \sum_{n=0}^{\infty} z^n, \quad |z| < 1$$

$$f_2(z) = \sum_{n=0}^{\infty} \frac{(z + i/2)^n}{(1 + i/2)^{n+1}}, \quad |z + \tfrac{i}{2}| < \tfrac{\sqrt{5}}{2}$$

$f_1 = f_2$ in the hashed region $\Rightarrow$ f_2 analytic continuation of f.

In fact, $f(z) = \frac{1}{1-z}$ extends f_1 & f_2 to whole complex plane (except singularity at $z = 1$).

Using such techniques, $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dz$, $\mathrm{Re}(z) > 0$, can be expanded to the whole complex plane except at $z = 0, -1, -2, \dots$

[a way to understand this is to take the rel. $\Gamma(z)\Gamma(1-z) = \pi / \sin \pi z$]

Similarly, $$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \mathrm{Re}\, z > 1$$

$$= \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^{t} - 1} dt, \quad \mathrm{Re}\, z > 1$$

can be extended to the whole complex plane, with only a simple pole at $z = 1$.

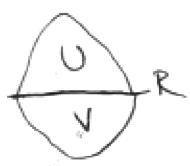
Example of a series that cannot be extended: $\sum_n z^{n!}$: singularities dense on unit circle.

<u>Morera's Theorem</u> If $\oint_C f(z)\,dz = 0$ for any closed path C inside U, then f analytic on U.

For, we can define $F(z) = \int_{z_a}^{z} f(z)\,dz$. F is differentiable on U

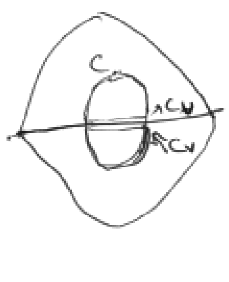
$\Rightarrow$ F analytic $\Rightarrow$ f analytic.

Now, say f is analytic in U, g analytic in V, and f continuous on $U \cup R$, g continuous on $V \cup R$, and $f = g$ on R (see fig)



Then we can analytically continue f & g to the unique analytic fn. $h(z)$ on $U \cup V \cup R$.

Proof:

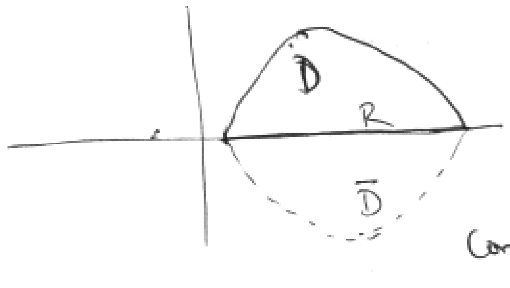


for any closed curve C,

$$\oint_C h(z)\,dz = \lim_{\varepsilon \to 0} \left[\oint_{C_U} h(z)\,dz + \oint_{C_V} h(z)\,dz \right] = 0$$

$\Rightarrow$ h analytic on $U \cup V \cup R$.

<u>Schwarz reflection principle</u>



Let $f(z)$ be analytic in D in upper half plane with boundary R on real line, and f real on real axis. Then we can ~~extend~~ define analytic continuation of f into $\bar{D}$, which is reflection of D about real axis, through: $g(z) = f^*(z^*)$ when $z \in \bar{D}$.

Proof: First we show g is analytic in $\bar{D}$

for a closed contour $\bar{C}$ in $\bar{D}$, parametrized by $z = \eta^*(t)$,

$$\oint_{\bar{C}} g(z)\,dz = \oint_{\bar{C}} g(\eta^*(t)) \frac{d\eta^*(t)}{dt}\,dt = \oint_{\bar{C}} f^*(\eta(t)) \frac{d\eta^*(t)}{dt}\,dt$$

$$= \left[\oint_C f(\eta(t)) \frac{d\eta}{dt}\,dt\right]^* = \oint_C f\,dz = 0$$

where c, parametrized by $z = \eta(t)$, is reflection of $\bar{C}$.

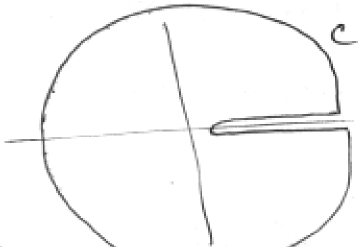
Now g is analytic on $\bar{D}$ and $f = g$ on the boundary (real axis)

$\Rightarrow$ g is the analytic continuation of f in $\bar{D}$.

$\Rightarrow$ if h analytic in a region including part of real axis, where it is real $\Rightarrow h(z^*) = h^*(z)$

Now say we have a function $h(z)$, analytic in $\mathbb{C}$ except a cut along real axis from x_0 to ∞. Say $h(z)$ is real on remainder of real axis, and $|h(z)| \to 0$ faster than $\frac{1}{z}$ as $|z| \to \infty$.

Then, taking the contour $\rightarrow$



$$h(z) = \frac{1}{2\pi i} \int_C \frac{h(z')}{z'-z}\,dz \quad , \text{ for } z \text{ not on real axis}$$

$$= \frac{1}{2\pi i}\left[\int_{x_0+i\varepsilon}^{\infty+i\varepsilon} \frac{h(z')}{z'-z}\,dz' - \int_{x_0-i\varepsilon}^{\infty-i\varepsilon} \frac{h(z')}{z'-z}\,dz'\right]$$

$$= \frac{1}{2\pi i} \int_{x_0}^{\infty} \left[\frac{h(x'+i\varepsilon)}{x'-z+i\varepsilon} - \frac{h(x'-i\varepsilon)}{x'-z-i\varepsilon}\right] dx'$$

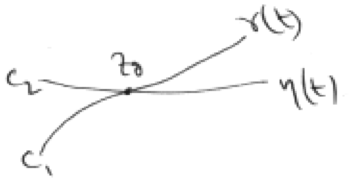
and, taking $\varepsilon \to 0$ (and remembering z not on real axis)

$$h(z) = \frac{1}{2\pi i} \int_{x_0}^{\infty} \lim_{\varepsilon\to 0^+} \frac{h(x'+i\varepsilon) - h(x'-i\varepsilon)}{x'-z}\,dx'$$

Now $h(x'-i\varepsilon) = h^*(x'+i\varepsilon) \Rightarrow h(z) = \frac{1}{\pi} \int_{x_0}^{\infty} \frac{\operatorname{Im} h(x'+i\varepsilon)_{\varepsilon\to 0^+}}{x'-z}\,dx'$

Conformal transformation:

A transform $z \to f(z)$ is angle-preserving (conformal) if $f(z)$ is analytic: if there are two curves in the complex plane, given parametrically by $\gamma(t)$ and $\eta(t)$, crossing at z_0, then the angle betw. tangents to the curves at z_0 is preserved by the transform [if $f'(z) \neq 0$].



Let the intersecting point $= \gamma(t_0)$ for C_1 and $\eta(t_1)$ for C_2. The tangent vectors $= \gamma'(t_0)$ and $\eta'(t_1)$.

Denoting the ~~conformal~~ map of the curves by C_1' and C_2', we can ~~write as~~ parametrize C_1' by $f(\gamma(t)) = f_\gamma(t)$ and $C_2' : f(\eta(t)) = f_\eta(t)$ crossing at $f(z_0)$

$\Rightarrow$ tangent vectors: $f'(z_0)\gamma'(t_0)$ and $f'(z_0)\eta'(t_1)$

$\Rightarrow$ angle remains same: ~~[for two complex~~ multiplication by $f'(z_0) = re^{i\theta}$ multiplies both tangent vectors by r and rotates them by θ.

Some examples: $z \to \frac{1}{z}$: maps the interior of unit circle to the outside and vice versa.

It also maps circles to circles: if z denotes points on a circle of radius r centred around z_0, $|z - z_0| = r \Rightarrow |z|^2 - 2\,\mathrm{Re}(\bar{z}_0 z) + |z_0|^2 - r^2$

mapped to $\left|\frac{1}{z'} - z_0\right| = r \Rightarrow |z'|^2 - 2\,\mathrm{Re}\left(\frac{\bar{z}_0}{|z_0|^2 - r^2} z'\right) + \frac{|z_0|^2}{(|z_0|^2 - r^2)^2} - \frac{r^2}{(|z_0|^2 - r^2)^2}$

$\Rightarrow$ circle centred at $\frac{z_0}{|z_0|^2 - r^2}$, radius $\frac{r}{|z_0|^2 - r^2}$

if $|z_0| = r$ (so the circle passes through origin): $2\,\mathrm{Re}(a z') - 1 = 0$

$\Rightarrow$ straight line.

transformations $z' = \frac{az+b}{cz+d}$: $\frac{dz'}{dz} = \frac{ad-bc}{(cz+d)^2}$

$\Rightarrow$ conformal transform. if $ad-bc \neq 0$ and $z \neq -\frac{d}{c}$

Can be thought of as combination of the maps $z_1 = z + \frac{d}{c}$, $z_2 = c^2 z_1$,

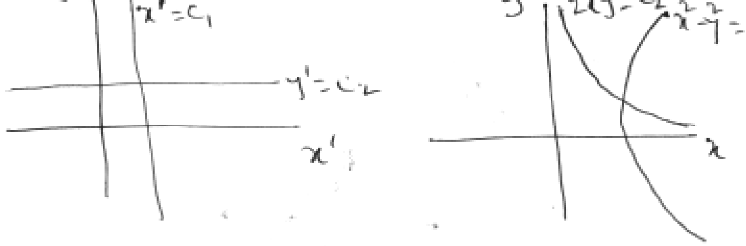
$z_3 = \frac{1}{z_2}$, $z_4 = (bc-ad)\, z_3$, $z' = \frac{a}{c} + z_4$.

$\Rightarrow$ maps circles to circles, or straight lines.

If we take $f(z) = z^2$ (double covering)

$$z' = x' + iy' = (x^2 - y^2) + i(2xy)$$

$\Rightarrow$ $x' = c_1$, $y' = c_2$ correspond to (orthogonal) hyperbolas in the xy-plane.



using C-R conditions, if $f(z) = u + iv$ analytic,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \text{harmonic fn}$$

Application: eg., for solving ~~electrostatic~~ problems where a fn.

$\varphi(x,y)$ satisfies Laplace's eqn. $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$

$\Rightarrow$ φ ~~thought~~ real part of an analytic fn. $f(z) = \varphi + iv$

imaginary part: $\frac{\partial v}{\partial y} = \frac{\partial \varphi}{\partial x} = g(x,y)$ $\Rightarrow$ $v = \int^y g(x,y')\,dy' + C(x)$

using $\frac{\partial v}{\partial x} = -\frac{\partial \varphi}{\partial y}$: $v = \int^y g(x,y')\,dy' + c$

$\Rightarrow$ Can find (locally) an analytic fn. corresponding to harmonic φ

Now $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\varphi + iv = f) = 4\frac{\partial^2 f}{\partial z\, \partial z^*} = 0$, if f analytic

If we do a conformal mapping $z = h(\omega)$, where $\omega = \xi + i\eta$, Then $\frac{\partial^2 f}{\partial w \partial w^*} = |h'(w)|^2 \frac{\partial^2 f}{\partial z \partial z^*}$

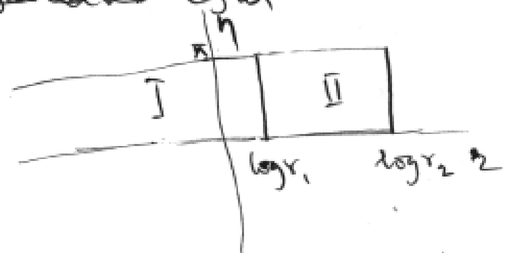
$\Rightarrow$ if φ satisfies Laplace's eqn. $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$, then $\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \eta^2} = 0$.

<u>Example</u> take two conducting coaxial cylinders, infinitely long, at potentials V_1 & V_2 resp.

Potential in the region betw. cylinders satisfies $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$

Taking $w = \log z$: ~~as the cylinders are...~~

$\xi = \log r$, $\eta = \theta$



V $\quad \frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} = 0$

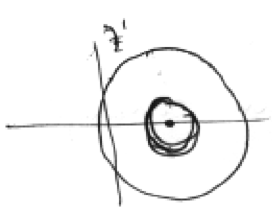
with boundary conditions: $V(\xi = \xi_1 = \log r_1) = V_1$, $V(\xi = \xi_2 = \log r_2) = V_2$

$\Rightarrow V = a\xi + b$ $\qquad V_1 = a\xi_1 + b$, $V_2 = a\xi_2 + b$

$\Rightarrow a = \frac{V_2 - V_1}{\xi_2 - \xi_1}$, $b = \frac{V_1 \xi_2 - V_2 \xi_1}{\xi_2 - \xi_1}$

$\Rightarrow V = a \ln r + b$, $\quad a = \frac{V_2 - V_1}{\ln r_2/r_1}$, $b = \frac{V_1 \ln r_2 - V_2 \ln r_1}{\ln \frac{r_2}{r_1}}$

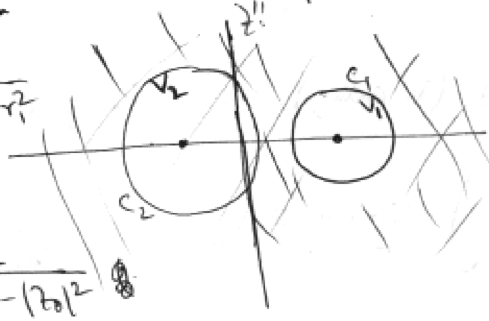
Now take $z' = z + z_0$:



$\Rightarrow V = a \ln |z' - z_0| + b$

$z'' = \frac{1}{z'}$: in z'' plane, the interior of 1st circle maps to exterior of circle

centred at $\frac{z_0}{|z_0|^2 - r_1^2}$, radius $\frac{r_1}{|z_0|^2 - r_1^2}$



2nd circle: $\frac{z_0}{|z_0|^2 - r_2^2}$, radius $\frac{r_2}{r_2^2 - |z_0|^2}$

$\Rightarrow$ renaming z'' as z, the soln. for this configuration given by

$$V = a \ln \left| \frac{1}{x+iy} - z_0 \right| + b$$

In particular, taking $z_0 = \sqrt{r_2 r_1}$ we get the potential when two cylinders of equal radius are kept at fixed potentials.

Method of steepest descent | saddle point

take an integral of the from $I(s) = \int_C e^{s f(z)}\, dz$

where $f(z)$ is analytic in a region including C. When s is large, it may be possible to deform the contour such that major contribution to I comes from a small part of the contour.

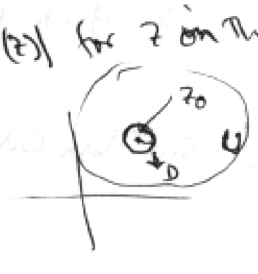
Deform contour to C', which has a region where $\mathrm{Im} f = v$ is nearly constant and u $(= \mathrm{Re} f)$ has a maximum.

Can u have a local maximum?

maximum modulus Theorem: if f analytic in U, and has a local maximum at $z_0 \in U$ so that in a neighborhood of z_0, $|f(z)| \le |f(z_0)|$, then $f(z)$ constant over U.

Proof: take a circle of radius ϵ, centre z_0 within this neighborhood. Then

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \epsilon e^{i\theta})\, d\theta \Rightarrow \text{if } |f(z_0)| \ge |f(z)| \text{ for } z \text{ on the circle, } f(z) = f(z_0).$$



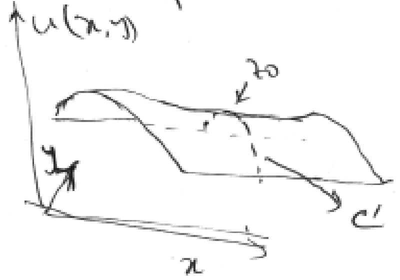
Considering any $\epsilon' < \epsilon$, we conclude $f(z) = f(z_0)$ everywhere inside the disc of radius ϵ.

$\Rightarrow f(z) = f(z_0)$ on D $\Rightarrow$ by analytic continuation $f(z) = f(z_0)$ on U.

Now if f is analytic, so is e^f. $|e^f| = e^u$ so u cannot have a maximum either. [also clear since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.]

What we are looking for is a maximum of u along the contour.

The point z_0 will be a saddle point of u rather than maximum.



Cauchy-Riemann condn $\Rightarrow \nabla u \cdot \nabla v = 0$

since $\vec{\nabla} v$ is perpendicular to v = constant line;

if c' has v = constant, it is the direction of $\vec{\nabla} u$ $\Rightarrow$ "steepest descent"

since major contribution to I (distorted to c') comes from the region around z_0, we write

$$f(z) = f(z_0) + \tfrac{1}{2}(z-z_0)^2 f''(z_0) + \cdots$$

~~If $f''(z_0) = re^{i\theta}$~~ we choose c' such that $\frac{1}{2}(z-z_0)^2 f''(z_0) = -\frac{1}{2}t^2$ with t real $\Rightarrow$ if $f''(z_0) = -re^{i\theta}$, $z - z_0 \approx \frac{t}{\sqrt{r}} e^{-i\theta/2}$

$$\Rightarrow \text{replace } I \text{ by } \int_{-\infty}^{\infty} e^{sf(z_0)} e^{-\frac{1}{2}st^2} \frac{dt}{\sqrt{r}} e^{-i\theta/2}$$

$$= \frac{e^{sf(z_0)}}{\sqrt{|f''(z_0)|}} \sqrt{\frac{2\pi}{s}} \cdot e^{-i\theta/2}$$

If we have $I = \int_c g(z) e^{sf(z)} dz$ and $g(z)$ is a slowly varying fn.,

a similar calculation will yield $I = g(z_0) \cdot e^{sf(z_0)} \cdot \sqrt{\frac{2\pi}{s|f''(z_0)|}} \cdot e^{-i\theta/2}$.

Example: The Gamma function has this integral repr.:

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt$$

valid for $Re(z) > 0$ (see, eg., Arfken; eq. 10.5 in 3rd edn.)

We can use this repr. and the saddle point method to find Stirling's formula, which is leading asymptotic behavior of $\Gamma(z)$ for $|z| \to \infty$

For notational simplicity let us start with $\Gamma(z+1) = z! = \int_0^\infty e^{-t}\, t^{z}\, dt$

We consider the complex integral $I(z) = \int_c e^{-\omega'}\, \omega'^{z}\, d\omega'$

$$= \int_c e^{z \ln \omega' - \omega'}\, d\omega'$$

variable transform. $\omega' = z\omega$ $\Rightarrow$ $I(z) = \int_c e^{z \ln z}\, e^{z(\ln \omega - \omega)}\, z\, d\omega$

$$= z^{z+1} \int_c e^{r f(\omega)}\, d\omega$$

where $f(\omega) = (\ln \omega - \omega)\, e^{i\theta}$ and $z = r e^{i\theta}$

$\Rightarrow$ saddle point at $\omega_0 = 1$, $f(\omega) \approx f(\omega_0) + \frac{1}{2}(\omega - \omega_0)^2 f''(\omega_0)$

$f''(\omega_0) = -e^{i\theta}$ $\Rightarrow$ $\omega \approx 1 + t e^{-i\theta/2}$

$$I(z) \approx z^{z+1} \cdot \int_{-\infty}^{\infty} e^{-re^{i\theta}}\, e^{-\frac{1}{2} r t^2}\, dt\, e^{-i\theta/2} = z^{z+1} \cdot e^{-z} \cdot \sqrt{\frac{2\pi}{r}} \cdot e^{-i\theta/2}$$

$$= z^{z}\, e^{-z} \cdot \sqrt{2\pi z}\, .$$