# Group Theory

Invertible linear transformn $f: V \to V$, eg., basis transformn, have these properties:

a) can be undone: $f^{-1} f |v\rangle = |v\rangle$

b) given $f_1, f_2$ : $f_1 f_2$ also defines a transformn.

c) $f_1 (f_2 f_3 |v\rangle) = (f_1 f_2) f_3 |v\rangle = f_1 f_2 f_3 |v\rangle$

d) can define "no transformn." $I$ : $I|v\rangle = |v\rangle \ \forall |v\rangle \Rightarrow I f|v\rangle = f|v\rangle$
$\forall f$

Example: rotn. on $\mathbb{R}^2$; reflection on $\mathbb{R}^2$
objects satisfying a) – d) are said to constitute a group.

Example of transformns that do not form group: projection operation.

<u>Defn.</u> Group: $G$ = a set of objects $\{a, b, c, \dots\}$ with a law of multiplication such that

$$\forall \ a, b \in G, \ ab \in G$$

$$\exists \ e \text{ s.t. } ea = a \ \forall a$$

$$\exists \ a^{-1} \ \forall a \ \text{ s.t. } a^{-1} a = e$$

$$\text{associativity: } (ab)c = a(bc)$$

Examples: i) set of integers under addition. $e = 0$, $a^{-1} = -a$. '$ab$' $= a+b = ba$
(set of +ve integers: do not form group)

ii) set of rational nos (excl. 0) under multiplication $ab = ba$

iii) Set of rotns in 2d ~~ab~~

iv) Set of rotns in 3d

v) Set $\{e, a, a^2\}$ with $a^3 = e$

If $ab = ba$, abelian group. Example: i)–iii) and v) above. iv) non-abelian.

If # elements is finite, ⇒ finite group. Only v) above is finite.
order of a group = # of elements.

<u>Some properties of inverse & identity:</u>

① right inverse exists and is equal to left inverse:

$$\text{if } a^{-1}a = e, \quad aa^{-1} = (a^{-1})^{-1}a^{-1}aa^{-1} = (a^{-1})^{-1}a^{-1} = e$$

② if $ea = a$, $ae = aa^{-1}a = ea = a$

③ if $ab = ac \Rightarrow b = c$

④ identity unique: if $ae = a, ae' = a \Rightarrow e = e'$

⑤ $(a^{-1})^{-1} = a$; $(ab)^{-1} = b^{-1}a^{-1}$.

for a finite group, any element $a$ must have the property that $a^r = e$ for some integer $r$ [otherwise completion of group requires $e, a, a^2, a^3, \ldots$ to be elements ⇒ infinite group]

If $a^r = e$, and $a^s \neq e$ for $0 < s < r$ ⇒ ~~[illegible]~~ $a$ is called an element of order $r$.

If order of an element = order of the group: group is $\{1, a, a^2, \cdots, a^{r-1}\}$

Such a group is called cyclic group.

Group of order 1: $\{e\}$

2: $\{e,a\}$, $a^2=e$ multiplication table:

| | e | a |
|---|---|---|
| e | e | a |
| a | a | e |

3: $\{e,a,b\}$ has to be cyclic: $b=a^2$

| | e | a | b |
|---|---|---|---|
| e | e | a | b |
| a | a | b | e |
| b | b | e | a |

[multiplication table: each element appears once in each row/column. If $a^2=e$, $ab=b$ $\Rightarrow a=e$ ✗]

4: $\{e,a,b,c\}$ one option: $\{e,a,a^2,a^3\}$ (call 4a)

another option: $a^2=b^2=c^2=e$, $ab=c$ etc (4b)

$\Rightarrow$

| | e | a | b | c |
|---|---|---|---|---|
| e | e | a | b | c |
| a | a | e | c | b |
| b | b | c | e | a |
| c | c | b | a | e |

If $H\subset G$ is a group under same multiplication law: subgroup

For any $G$, $\{e\}$ and $G$ are subgroups. Any other subgroup is called a proper subgroup.

eg. for (4a), $\{e,a^2\}$ proper subgroup. (4b): $\{e,a\}$ proper subgroup

Now say $H$ is proper subgroup of $G$. Take $g\in G$, $g\notin H$.

$gH$ is disjoint from $H$: if $gh_i=h_j$, $g=h_jh_i^{-1}\in H$.

Take $g'\notin H, gH$ Then $g'H$ disjoint from $H, gH$

This way: $G=H+gH+g'H+\cdots$ so $n_G=m\cdot n_H$.

$\Rightarrow$ order of a subgroup H is a factor of order of the group.

The sets $H, gH, g'H, \ldots$ are called left cosets of H in G.

Similarly we could construct right cosets, which are different in general.

Since in a group, $\{e, a, a^2, \ldots, a^{r-1}\}$ always forms a subgroup

$\Rightarrow$ order of the group is a multiple of order of any element.

$\Rightarrow$ if order of group = prime number, the group is cyclic.

If $aH = Ha$ $\forall a \in G$, H called an invariant subgroup of G

$h_i a = a h_j \Rightarrow h_i = a h_j a^{-1}$ $\forall a \in G$

If G has no invariant subgroup $\Rightarrow$ simple.

no abelian invariant subgroup $\Rightarrow$ semisimple.

If $gag^{-1} = b$, where $a, g, b \in G$, a is said to be conjugate to b.

all elements formed doing $gag^{-1}$ are said to form a conjugacy class.

For abelian groups, $gag^{-1} = a$, $\Rightarrow$ each element forms a conjugacy class.

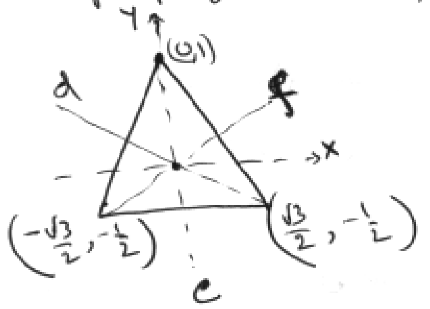
If H is an invariant subgroup of G, the cosets gH form a group with a multiplication law determined by that of G:

$aH \cdot bH = abHb^{-1}bH = abH$. Identity $= H$.

This is called the quotient group $G/H$. order $= \frac{n_G}{n_H}$.

Ex: $G = \{e, a, a^2, a^3\}$, $H = \{e, a^2\}$ $\quad G/H = \{E, A\} = \mathbb{Z}_2$

Ex: Group of order six: symmetries of an equilateral triangle




symmetries: along z-axis, rotns by $\frac{2\pi}{3}$, $\frac{4\pi}{3}$
$\qquad a \qquad b$

reflections about the three axes: $c, d, f$

multiplication table:

| | e | a | b | c | d | f |
|---|---|---|---|---|---|---|
| e | e | a | b | c | d | f |
| a | a | b | e | d | f | c |
| b | b | e | a | f | c | d |
| c | c | f | d | e | b | a |
| d | d | c | f | a | e | b |
| f | f | d | c | b | a | e |

this is nonabelian (smallest non abelian group).
(**This** group is known as $C_{3v}$).

$H = \{e, a, b\}$: abelian invariant subgroup

quotient group: $\{H, C_1\}$ where $C_1 = cH$.

Conjugacy classes: $\{e\}$, $\{a, b\}$, $\{c, d, f\}$.

Homomorphism If $G$, $H$ are two groups, homomorphism $\phi: G \to H$ is a mapping that preserves the group multiplication structure: $\phi(g_1)\cdot\phi(g_2) = \phi(g_1 g_2)$

Kernel of $\varphi$ : $\{g_i\}$ s.t. $\varphi(g_i) = e_H$. <u>Check</u> $\mathrm{Ker}(\varphi)$ is invariant subgroup of $G$, and $\mathrm{Im}(\varphi)$ subgroup of $H$.

image of $\varphi$ : all elements $\varphi(g_i)$.

If $\mathrm{Im}(\varphi) = H$, the mapping is onto.

If $\varphi$ is one-one and onto, it is called isomorphism.

<u>Check</u> if mapping is isomorphism, $\mathrm{Ker}(\varphi) = \{e\}$.

<u>Representation</u> homomorphism ~~of matrices GL(n)~~

$G \to n \times n$ matrices $GL(n)$. Called $n$-dim repr.

If $g$ mapped to $D(g) \Rightarrow D(e) = \mathbb{1}$, $D(g^{-1}) = D^{-1}(g)$,

$D(g_1 g_2) = D(g_1) \cdot D(g_2)$ etc.

If mapping one-one, it is called a faithful repr.

<u>Example</u> For $C_{3v}$, from the coordinates, an obvious 2-dim faithful repr.

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} -1/2 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -1/2 \end{pmatrix}, \quad b = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix},$$

$$c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad f = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

An unfaithful repr. : all elements $\to \mathbb{1}$

this is called the trivial repr.

Of course, by a change of basis, the repr. can be changed to $S^{-1} D(g) S \quad \forall g$. Reprs connected like this are called equivalent repr.

<u>Theorem</u> Any repr. of a finite group is equivalent to a unitary representation.

<u>Proof</u> : Say $D$ not unitary $\Rightarrow \langle Dx | Dy \rangle \neq \langle x | y \rangle$

This means columns of $D$ are not orthonormal (under normal inner product $\langle x|y\rangle = \sum_i x_i^* y_i$)

Define a new inner product: $(x|y) = \frac{1}{n_G} \sum_{g \in G} \langle D(g)x | D(g)y \rangle$

Check $(x|y)$ is a valid inner product.

$\Rightarrow$ for any two vectors $|x\rangle, |y\rangle$ : $(D(h)x | D(h)y) = (x|y)$

$\Rightarrow$ $D$ unitary in this inner product

Now say $|e_i\rangle$ is orthonormal basis : $\langle e_i | e_j \rangle = \delta_{ij}$

and $|e_i'\rangle$ is orthonormal basis under the new inner product: $(e_i'|e_j') = \delta_{ij}$

Expand: $|e_i'\rangle = T_{ij} |e_j\rangle$ . Form $\tilde{D}(g) = T^{-1} D(g) T$.

Then $\tilde{D}(g)$ is the required unitary reprn.

For, $\langle \tilde{D}(g)x | \tilde{D}(g)y \rangle = \langle T^{-1} D(g) T x | T^{-1} D(g) T y \rangle$

$= (D(g) Tx | D(g) Ty) = (Tx|Ty) = \langle x|y \rangle$

<u>Reducible and irreducible reprn</u>

If we have reprn. of $G$ on a $n$-dim vector space, but the action of elements of $G$ is such that a subspace of the vector space is invariant under $G$, we call it reducible reprn.

If the reprn is reducible, say $W$ is $m$-dim subspace of $n$-dim. $V$ which is left invariant by $G$. Choosing a basis in $V$ such that first $m$ vectors span $W$: clearly all $D(g)$ are of the form $\begin{pmatrix} A_{m\times m} & B_{m\times(n-m)} \\ 0 & C_{(n-m)\times(n-m)} \end{pmatrix}$.

If $D(g)$ is unitary, the form becomes $\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$.

If a reprn. is not reducible, it is called irreducible (or irrep).

<u>Schur's lemma</u> ① If $D(g)$ and $D'(g)$ two reprns, dim. $n$ & $n'$ with $n > n'$, $D(g)$ reducible iff $\exists A$ such that $D(g) A = A D'(g)$.

<u>Proof</u> D is reprn $\Rightarrow$ $|v\rangle \to D|v\rangle$

$v_i \to D_{ij} v_j$

If D reducible, $\exists$ a set $|v'\rangle = A|v\rangle$, $v'_i = A_{ij} v_j$

such that $v'_i \to D'_{ij} v'_j = D'_{ij} A_{jk} v_k$

also $v'_i = A_{ij} v_j \to A_{ij} D_{jk} v_k$ $\Rightarrow D'_{ij} A_{jk} v_k = A_{ij} D_{jk} v_k$

Since this is true for all $|v\rangle$ : $D'A = AD$.

Easy to see that the argument can be run backwards $\Rightarrow$ if $D'A = AD$ then D is reducible.

1a If $D$, $D'$ are irreps, with $AD = D'A$, then either $n = n'$ and $A$ is invertible, or $A = 0$.

For, if $n > n'$, from the argument above, D is reducible

If $n' > n$, $D'$ is obviously reducible.

If $n = n'$, if A is invertible then $D$ & $D'$ are equivalent.

If A not invertible, $\det A = 0 \Rightarrow |v'\rangle = A|v\rangle$ span a lower dimensional space than that spanned by $|v\rangle$. Since this space is invariant under $D'$, the n-dim reprn. is reducible.

<u>Schur's lemma</u> ② If $AD(g) = D(g)A \ \forall g \in G$ and $D(g)$ is irrep, then

$A = \lambda I$.

<u>Proof</u> take $A|v\rangle = \lambda|v\rangle$

Now $D(g)A|v\rangle = \lambda D(g)|v\rangle = A D(g)|v\rangle \Rightarrow D(g)|v\rangle$ is

eigenvector with eigenvalue $\lambda$

Since it is irrep, all $D(g)|v\rangle$ span the space $\Rightarrow A = \lambda I$ in

this space.

<u>Example</u> Take the 2-dim irrep of $C_{3v}$.

Simpler to work in the basis such that $a$ is diagonal:

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \qquad S^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$\tilde{a} = S^{-1}aS = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad \omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \omega^2 = e^{4\pi i/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\tilde{b} = S^{-1}bS = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \quad \tilde{c} = S^{-1}cS = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \tilde{d} = S^{-1}dS = i\begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$$

$$\tilde{f} = S^{-1}fS = i\begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ commute with all.

$$A\tilde{a} = \tilde{a}A \Rightarrow \begin{pmatrix} a\omega & b\omega^2 \\ c\omega & d\omega^2 \end{pmatrix} = \begin{pmatrix} a\omega & b\omega \\ c\omega^2 & d\omega^2 \end{pmatrix} \Rightarrow b = c = 0 \Rightarrow \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

$$a\tilde{c} = \begin{pmatrix} 0 & a \\ -d & 0 \end{pmatrix} = \tilde{c}a = \begin{pmatrix} 0 & d \\ -a & 0 \end{pmatrix} \Rightarrow a = d \Rightarrow A = a\mathbb{1}.$$

<u>Corollary</u> for abelian group, all irreps are 1-dim. Since if not, since $D(h)$ commutes with all $D(g)$, $D(h) = a \cdot \mathbb{1} \Rightarrow$ reducible.

<u>Example</u> $\{e, a, b\}$ is abelian in above example $\Rightarrow$ fully reducible.

Now take $A = \sum_g D(g) X D(g^{-1})$, $X$: $n_r \times n_r$ matrix and $D(g)$: $n_r$ dim. irrep

$$\Rightarrow A\, D(g) = D(g)\, A \Rightarrow A = \lambda I$$

Take $X$: $jk$th element $=1$, rest $=0$

$$\Rightarrow \sum_g D(g)_{ij}\, D(g^{-1})_{kl} = \lambda\, \delta_{il}$$

Taking $i=l$, sum $\Rightarrow \sum_g \sum_i D(g)_{ij}\, D(g^{-1})_{ki} = \sum_g \delta_{kj} = n_G\, \delta_{jk} = \lambda\, n_r$

$$\Rightarrow \lambda = \frac{n_G}{n_r}\delta_{jk}$$

$$\Rightarrow \sum_g D(g)_{ij}\, D(g^{-1})_{kl} = \frac{n_G}{n_r}\delta_{jk}\,\delta_{il}$$

If we take inequivalent irreps $r, s$: $A=0$ so combining,

$$\sum_g D^r(g)_{ij}\, D^s(g^{-1})_{kl} = \frac{n_G}{n_r}\delta_{rs}\,\delta_{jk}\,\delta_{il}$$

If unitary repr.: $\sum_g D^r(g)_{ij}\, D^s(g)^*_{lk} = \frac{n_G}{n_r}\delta_{rs}\,\delta_{jk}\,\delta_{il}$

Think of $D^r(g)_{ij}$ as the $g$th component of a vector in an $n_G$-dim vector space $\Rightarrow$ the above relation indicates $\sum_{\text{all irreps } r} n_r^2$ orthogonal vectors in this space. $\Rightarrow \sum_{\text{irreps } r} n_r^2 \leq n_G$.

<u>Characters</u> The explicit matrices $D(g)$ have superfluous information → e.g. two equivalent repr.s have different $D(g)$. We can, instead, work with $\mathrm{tr}\, D(g)$, called character, $\chi(g)$. This does not have complete information: elements in same conjugacy class have same character (so better to talk of $\chi(C)$, character of a conjugacy class).

From the above orthogonality relation, $\sum_g \chi^r(g)\, \chi^s(g^{-1}) = n_G\, \delta_{rs}$

for unitary : $\sum_g \chi^r(g)\, \chi^s(g)^* = n_G\, \delta_{rs}$

if $n_c$ : no. of elements in conjugacy class $c$,

$$\Rightarrow \sum_c n_c\, \chi^r(c)\, \chi^s(c)^* = n_G\, \delta_{rs}$$

using similar arguments ($\sqrt{n_c}\,\chi^r(c)$ being $c$th component of a vector

$\Rightarrow$ $n_{irrep}$ orthogonal vectors where $n_{irrep}$ = total # of inequivalent irreps of $G$) $\Rightarrow$ no. of inequivalent irreps $\leq$ # of conjugacy classes

Now take a (possibly reducible) rep $D$ in unitary rep, and use a basis such that $D(g) = \bigoplus_r a_r D^r(g)$, The sum is over irreps and $a_r$: integer

$$\Rightarrow \chi_c = \sum_r a_r \chi_c^{(r)}$$

$$\Rightarrow \sum_c \chi_c^{s*} \chi_c n_c = \sum_r a_r \sum_c \chi^{s*}_c \chi^r_c n_c = n_G\, a_s$$

$$\Rightarrow a_r = \frac{1}{n_G} \sum_c \chi^r(c)^* \chi(c)\, n_c .$$

Using this, we can show that $\sum_{\text{irreps } r} n_r^2 = n_G$. for this, we use a {reducible} $n_G \times n_G$ representation of $G$ that follows directly from the group multiplication table : if in the table, the row against $a_i$ is $a_{i_1} \cdots a_{i_{n_G}}$ (i.e. $a_i . (a_1 \cdots a_{n_G}) = a_{i_1} \cdots a_{i_{n_G}}$) then take

$$D(a_i)_{jk} = \delta_{k, i_j}$$

Example : from our multiplication table for $C_{3v}$, $D(e) = \mathbb{1}_6$,

$$D(a) = \begin{pmatrix} 0&1&0&&& \\ 0&0&1&&0& \\ 1&0&0&&& \\ &&&0&1&0 \\ &0&&0&0&1 \\ &&&1&0&0 \end{pmatrix}, D(b) = \begin{pmatrix} 0&0&1&&& \\ 1&0&0&&0& \\ 0&1&0&&& \\ &&&0&0&1 \\ &0&&1&0&0 \\ &&&0&1&0 \end{pmatrix}, D(c) = \begin{pmatrix} &&&1&0&0 \\ &0&&0&0&1 \\ &&&0&1&0 \\ 1&0&0&&& \\ 0&0&1&&0& \\ 0&1&0&&& \end{pmatrix}, D(d) = \begin{pmatrix} &&&0&1&0 \\ &0&&1&0&0 \\ &&&0&0&1 \\ 0&1&0&&& \\ 1&0&0&&0& \\ 0&0&1&&& \end{pmatrix}, D(f) = \begin{pmatrix} &&&0&0&1 \\ &0&&0&1&0 \\ &&&1&0&0 \\ 0&0&1&&& \\ 0&1&0&&0& \\ 1&0&0&&& \end{pmatrix}$$

In this repn., $D(e) = \mathbb{1}_{n_G}$ so $\chi(e) = n_G$, $\chi(g \neq e) = 0$

Therefore $\sum_c \chi(c)^* \chi(c)\, n_c = n_G^2$

Now if this reprn. has irrep $r$ with multiplicity $a_r$, etc,

then $a_r = \frac{1}{n_G} \sum_c \chi^{r*}(c)\, \chi(c)\, n_c = \frac{1}{n_G} \cdot n_r \cdot n_G = n_r$

also $\chi_c = \sum_r a_r \chi^r(c) \Rightarrow n_G^2 = \sum_c \sum_r a_r \chi^r(c)^* \chi(c)\, n_c$

$$= \sum_r a_r \cdot n_G a_r = n_G \sum a_r^2$$

$$\Rightarrow n_G = \sum_r n_r^2 .$$

So, eg., $C_{3v}$ has 1, 2-dim irrep and 2, 1-dim irreps.

We can construct a "character table" of the irreps.

| | $C_1 = \{e\}$ | $C_2 = \{a, b\}$ | $C_3 = \{c, d, f\}$ |
|---|---|---|---|
| $D^1_{(1\text{-dim})}$ | 1 | 1 | 1 |
| $D^2_{(1\text{-dim})}$ | 1 | 1 | $-1$ |
| $D^3_{(2\text{-dim})}$ | 2 | $-1$ | 0 |

Can be constructed by noting: ① 1st row (trivial reprn) and 1st column ($\chi$ for identity) are obvious.

② use $\sum_c n_c \chi^r(c) \chi^s(c)^* = n_G \delta_{rs}$

③ $C_3$: reflections. So $D(C_3^2) = 1$. $\Rightarrow \chi_{1\text{-dim rep}}(C_3) = \pm 1$. Similarly

$\chi_{1\text{-dim}}(C_2) = 1, e^{2\pi i/3}, e^{-2\pi i/3}$.

$\Rightarrow$ gives 2nd row.

Now for 3rd row, 2 orthogonality conditions $\Rightarrow$ determine the row.

Could also use restrictions like in ③ on 3rd row.

Of course, here we already know the 2-d reprn. So could write the third row straight.

Note that the columns are orthogonal. This can be shown to be true in general: $\sum_{\mu} \chi^{\mu}(C_i)\,\chi^{\mu}(C_j)^* = \delta_{ij} \frac{n_G}{n_{C_i}}$

$\Rightarrow$ by similar arguments, # conjugacy classes $\leq$ # ineq. irreps

$\Rightarrow$ # conjugacy classes = # ineq. irreps.

## Symmetry and quantum mechanics

One application of group theory is in finding the degeneracie of the energy eigenstates.

If system invariant under a symmetry group $R$,

take a unitary repr. $U_R$ : $[U_R, H] = 0$

If $|\psi\rangle$ energy eigenstate: $H|\psi\rangle = E|\psi\rangle \Rightarrow H\,U_R|\psi\rangle = E\,U_R|\psi\rangle$

Take a $d$-dim irrep $D(U_R)$ : $d$-fold degenerate eigenstate

[ $HD = DH \Rightarrow H = E\mathbb{1}$ in this subspace ].

eg. take a molecule like $BCl_3$, which has $C_{3v}$ symmetry.

$\Rightarrow$ ground state, parity odd state, 2-dim irrep.

Continuous groups Take, eg., $GL(2,\mathbb{R})$ : matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d$ real and $ad-bc \neq 0$

Non-abelian group with infinite elements $\Rightarrow$ the previous analyses not of much help.

Take an element close to unity: $A = \mathbb{1} + \delta a_i T_i$, where $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ etc.

This is example of a four-parameter continuous group. The $T_i$ are called generators of this group.

The group composition law $M(a_1 \cdots a_4) \cdot M(b_1 \cdots b_4) = M(c_1 \cdots c_4)$ give $c_i = \varphi_i(a, b)$. If $\varphi_i$ differentiable $\Rightarrow$ Lie group. $GL(2,\mathbb{R})$ is a Lie group.

If we expand $[T_i, T_j] = f_{ijk} T_k \Rightarrow f_{ijk}$ are called structure constants. [only for nonabelian group]

<u>Some examples</u> ① $SO(3)$ : 3-parameter group. Generators :

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2, T_3 \quad \text{with } [T_i, T_j] = \varepsilon_{ijk} T_k .$$

② $SU(N)$ : $U^+ U = \mathbb{1}$, $\det U = \mathbb{1}$. $(N^2 - 1)$ parameter group.