Vector space : generalizn. of / abstract version of Cartesian n-component vectors

$\vec{x} = (x_1 \cdots x_n)$, $\vec{y} = (y_1 \cdots y_n)$ $\Rightarrow$ $\vec{x}+\vec{y} = (x_1+y_1, \cdots, x_n+y_n)$ vector

$c\vec{x} = (cx_1 \cdots cx_n)$ vector

Our generalizn. would allow the set $\{x_i\}$ ~~of numbers~~ to be complex. Same with c.

More generally : define "field" F : set of 'numbers' $c_1 c_2 \cdots$ with

a) addition : if $c_1, c_2 \in F$, then $c_1+c_2 \in F$

$$c_1+c_2 = c_2+c_1$$

$$c_1+(c_2+c_3) = (c_1+c_2)+c_3$$

$\exists\ 0$ s.t. $x+0 = x \quad \forall x$

for each x $\exists -x$ s.t. $x+(-x) = 0$

b) multiplication : if $c_1, c_2 \in F$, $c_1c_2 \in F$

$$c_1c_2 = c_2c_1$$

$$c_1(c_2c_3) = (c_1c_2)c_3$$

$\exists\ 1 \neq 0$ s.t. $x.1 = x$

~~$\exists x \neq 0$~~
for each $x \neq 0$ $\exists x^{-1}$ s.t. $xx^{-1} = 1$

$$x(y+z) = xy+xz$$

Example : $\mathbb{R}$; $\mathbb{C}$; set of rational numbers

Set of integers $\mathbb{Z}$ not a field

Nontrivial example : set $(0,1)$ under addition mod 2 and multiplication.

Note : 0 unique : $0+0' = 0' = 0$. $-x$ unique : $x+x' = 0$
add $-x \Rightarrow x' = -x$.

<u>Vector space</u>

a) field F of scalars

b) set V of objects called ~~scalars~~ vectors

c) an addition s.t. if $|x\rangle, |y\rangle \in V$, $|x\rangle + |y\rangle \in V$

$$|x\rangle + |y\rangle = |y\rangle + |x\rangle$$

$$|x\rangle + (|y\rangle + |z\rangle) = (|x\rangle + |y\rangle) + |z\rangle$$

$$\exists\, |0\rangle \in V \text{ s.t. } |0\rangle + |x\rangle = |x\rangle \ \forall |x\rangle$$

$$\exists\, |-x\rangle \text{ s.t. } |x\rangle + |-x\rangle = |0\rangle$$

d) scalar multiplication: if $c \in F$ and $|x\rangle \in V$, $c|x\rangle \in V$

$$1|x\rangle = |x\rangle$$

$$(c_1 c_2)|x\rangle = c_1 (c_2 |x\rangle)$$

$$c(|x\rangle + |y\rangle) = c|x\rangle + c|y\rangle$$

$$(c_1 + c_2)|x\rangle = c_1|x\rangle + c_2|x\rangle$$

loosely: **V** vector space over field **F**.

<u>Example</u>: For any field F, ~~form~~ all n-tuples $|x\rangle = (x_1 \cdots x_n)$ with addition

$$|x\rangle + |y\rangle = (x_1 + y_1, \cdots, x_n + y_n)$$

and $c|x\rangle = (cx_1 \cdots cx_n)$

$\mathbb{R}^n$, $\mathbb{C}^n$

<u>Example</u> $\mathbb{C}^n$ also vector space over $\mathbb{R}$

but $\mathbb{R}^n$ not vector space over $\mathbb{C}$

<u>Example</u> Space of $F^{m\times n}$ matrices with usual matrix properties

<u>Example</u> V: set of all fns from a non-empty set S to F, with

$$(f+g)(s) = f(s) + g(s), \quad (cf)(s) = cf(s)$$

<u>Example</u> Set of all polynomial fns $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$

Note: associativity & commutativity $\Rightarrow$ $(|x\rangle + |y\rangle) + (|z\rangle + |w\rangle) = (|x\rangle + |z\rangle) + (|y\rangle + |w\rangle)$

$= |x\rangle + |y\rangle + |z\rangle + |w\rangle$

$\Rightarrow$ we'll talk about $\sum_i c_i |x_i\rangle$

Some properties $\quad c|0\rangle = |0\rangle \qquad c|0\rangle = c(|0\rangle + |0\rangle) = c|0\rangle + c|0\rangle$

add $-c|0\rangle \Rightarrow c|0\rangle = |0\rangle$

Similarly $0|x\rangle = |0\rangle$

If $c|x\rangle = |0\rangle$, either $c=0$, or $|x\rangle = |0\rangle$

if $c \neq 0$, $\quad c^{-1}c|x\rangle = c^{-1}|0\rangle = |0\rangle = |x\rangle$.

for all $|x\rangle$, $\quad -1|x\rangle = |-x\rangle$

Subspace $\quad$ if V vector space over F, a subset W of V is called a subspace if W is a vector space over F.

A non-empty subset W is subspace of V iff for any $|x\rangle, |y\rangle \in W$ and any $c \in F$, $\quad c|x\rangle + |y\rangle \in W$.

Example $\quad V$ and $\{|0\rangle\}$ are subspaces of V.

in F^n, $\quad$ n-tuples with, eg., $x_1 = 0$ is a subspace.

Space of all polynomial fns subspace of space of all fns.

Symmetric matrices $\Rightarrow$ subspace of space of all matrices.

If W_1, W_2 subspaces of V, so is $W_1 \cap W_2$.

If ~~S is a non-empty~~ $S = \{|s_1\rangle, \cdots, |s_n\rangle\}$ a non-empty subset of V, ~~then~~ the subspace of all combinations $\sum_i c_i|s_i\rangle$ is called subspace spanned by S.

The set S is called linearly dependent if there are vectors $|s_1\rangle \cdots |s_m\rangle$ in S such that $c_1|s_1\rangle + \cdots + c_m|s_m\rangle = 0$ with not all of $c_1 \cdots c_m$ zero. [Note: in such a case at least one of $|s_i\rangle$ can be written as linear combn. of others]

If for any subset $|s_1\rangle \cdots |s_m\rangle$, $c_1|s_1\rangle + \cdots + c_m|s_m\rangle = 0$ implies $c_1 = \cdots = c_m = 0$, S is linearly indep. set.

Example $\{e_1, e_2, e_3\}$ linearly independent set over $\mathbb{R}$.

$\{1, i\}$ l.i. if V defined over $\mathbb{R}$, not if V defined over $\mathbb{Z}$.

$\{1, x, x^2, \cdots, x^n\}$ linearly independent subset of the space of polynomials of order n, P_n

$\exists$ a maximal set of LI vectors $|e_i\rangle$: any vector $|V\rangle = \sum v_i |e_i\rangle$

Proof: $a_0|V\rangle + \sum_{i=1} a_i |e_i\rangle = 0$ with nonzero a_0.

Defn. Let V be a vector space. A basis for V is a linearly independent set of vectors in V which spans the space V.

Writing $|x\rangle = \sum x_i |e_i\rangle$ where $|e_i\rangle$ is a basis:

The decomposition is unique since if $|x\rangle = \sum x_i' |e_i\rangle$

$$\sum_i (x_i' - x_i)|e_i\rangle = 0 \implies x_i' = x_i \;\forall i.$$

~~arbitrary vector can be expanded~~

Theorem If V is spanned by $\{|x_1\rangle \cdots |x_m\rangle\}$, any LI set in V has no more than m elements.

Proof If we take $S = \{|y_1\rangle \cdots |y_n\rangle\}_{n>m}$ and decompose

$$|y_i\rangle = A_{ij}|x_j\rangle$$

Now $\sum_i c_i |y_i\rangle = \sum_{ij} A_{ij} c_i |x_j\rangle = 0$ if $\sum_i A_{ij} c_i = 0$

This is m constraints on n unknowns $c_i \implies \exists$ nontrivial soln. c_i

$\implies S$ is linearly dependent set.

$\implies$ any two bases of V have same no. of elements
$\implies$ dimension of V.

If $n = \dim V$, a) a subset with more than n elements is not LI

b) a subset with less than n elements cannot span V.

$\implies$ allows mapping any vector in a n-dim vector space over $\mathbb{R}$ (F) to $\mathbb{R}^n$ (F^n). [Need ordered basis.]

Change of basis: Let V is n-dimensional, with two bases $|e_i\rangle$ and $|e_i'\rangle$.

Say $|x\rangle = \sum x_i |e_i\rangle$. Expand $|e_i'\rangle = \sum_j P_{ji} |e_j\rangle$

$$\Rightarrow |x\rangle = \sum_i x_i' |e_i'\rangle = \sum_{ij} x_i' P_{ji} |e_j\rangle = \sum_j x_j |e_j\rangle$$

$$\Rightarrow x_j = \sum_i P_{ji} x_i' \qquad \text{matrix notn } \vec{x} = P \vec{x}'$$

If Q defines inverse transformn., $|e_i\rangle = \sum_j Q_{ji} |e_j'\rangle \Rightarrow x_j' = Q_{ji} x_i$

uniqueness of expansion: $P_{ji} Q_{ik} = \delta_{jk} \Rightarrow Q = P^{-1}$

Examples of subspace & basis: $\mathbb{R}^3$, $\mathbb{R}^2$ etc

Space of polynomials with degree $< n$ (n-dim subspace)

subspace: polynomials with degree $< k < n$. LI basis: $\{1, x, x^2, \cdots, x^n\}$

Infinite dimensional: set of all polynomials.

<u>Inner product / Scalar product / dot product</u>

Associate with $|x\rangle, |y\rangle \in V$ a number $c \in F$ such that

i) $\langle z | \alpha x + y\rangle = \alpha \langle z | x\rangle + \langle z | y\rangle$

ii) $\langle x | y\rangle = \langle y | x\rangle^*$

iii) $\langle x | x\rangle > 0$ if $|x\rangle \neq |0\rangle$

Note: $\langle x | 0\rangle = \langle x | y - y\rangle = \langle x|y\rangle - \langle x|y\rangle = 0 \quad \Rightarrow \langle 0 | x\rangle = 0$

Norm of a vector defined as $\|x\|$: real no. > 0 (unless $|x\rangle = |0\rangle$)

such that $\|cx\| = |c|\|x\|$ and $\|x+y\| \leq \|x\| + \|y\|$.

$\sqrt{\langle x|x\rangle}$ defines a norm. [Called inner product norm] $\Rightarrow$ we denote as $\|x\|$.

V is called inner product space if inner product defined.

if F real and V is finite dimensional ⇒ Euclidean space.

Properties : i) $|\langle x|y\rangle| \leq \|x\|\cdot\|y\|$ Cauchy-Schwartz.

Proof: Take $|z\rangle = |y\rangle - \frac{\langle x|y\rangle}{\langle x|x\rangle}|x\rangle$

$$0 < \langle z|z\rangle = \left(\langle y| - \frac{\langle y|x\rangle}{\langle x|x\rangle}\langle x|\right)\left(|y\rangle - \frac{\langle x|y\rangle}{\langle x|x\rangle}|x\rangle\right)$$

$$= \langle y|y\rangle - 2\frac{|\langle x|y\rangle|^2}{\langle x|x\rangle} + \frac{|\langle x|y\rangle|^2}{|\langle x|x\rangle|^2}\langle x|x\rangle = \langle y|y\rangle - \frac{|\langle x|y\rangle|^2}{\langle x|x\rangle}$$

Equality if $|y\rangle = c|x\rangle$ or $|x\rangle = |0\rangle$.

ii) $\|x+y\| \leq \|x\| + \|y\|$ Triangle inequality

$$\|x+y\|^2 = \langle x+y|x+y\rangle = \langle x|x\rangle + \langle y|y\rangle + \langle x|y\rangle + \langle y|x\rangle$$

$$\leq \langle x|x\rangle + \langle y|y\rangle + 2|\langle x|y\rangle|$$

$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\cdot\|y\| = (\|x\|+\|y\|)^2.$$

Metric space: distance $d(x,y)$ defined betw. two objects x,y such that

$d(x,y) > 0$ (except $d(x,x) = 0$), $d(x,y) = d(y,x)$, and $d(x,z) \leq d(x,y) + d(y,z)$

So inner product space is a metric space.

Other possibilities for metric: eg. $d(x,y) = 0$ if $x = y$, $= 1$ if $x \neq y$ forms a metric

Inner product: examples:

on F^n: $\langle x|y\rangle = \sum_i x_i^* y_i$ defines an inner product

⊛ variations possible; eg., on $\mathbb{R}^2$, $\langle x|y\rangle = x_1y_1 - x_2y_1 - x_1y_2 + 2x_2y_2$ defines an inner product.

in the space of all continuous complex-valued fns on $[a,b]$: $\langle f|g\rangle = \int_a^b dt\, f^*(t)\, g(t)$

~~forms~~ inner product

Say in W, we have an inner product $\langle w_1|w_2\rangle$;

V is same dim. as W, $T: V \to W$ maps V to W, T nonsingular.

Then $\langle v_1|v_2\rangle = \langle Tv_1|Tv_2\rangle$ defines an inner product on V.

If V spanned by $|e_i\rangle$, inner product ~~defined~~ specified by $g_{ij} = \langle e_i|e_j\rangle$

$$\langle x|y\rangle = \sum_{ij} g_{ij}\, x_i^* y_j$$

$\langle e_i|e_j\rangle = \langle e_j|e_i\rangle^* \Rightarrow g_{ij}$ hermitian; g_{ij} satisfies positivity: $g_{ij} x_i^* x_j > 0$

⊛ $\Rightarrow g_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

Orthogonal vectors: $\langle x|y\rangle = 0$.

Orthogonal basis: $\langle e_i|e_j\rangle = h_i \delta_{ij}$. Orthonormal basis: $\langle e_i|e_j\rangle = \delta_{ij}$

given a basis $\{|\alpha_i\rangle\}$, can construct orthonormal basis $\{|e_i\rangle\}$:

construct orthogonal set $|e_i'\rangle = |\alpha_i\rangle - \sum_{j<i} \frac{\langle e_j'|\alpha_i\rangle}{\langle e_j'|e_j'\rangle} |e_j'\rangle$

$|e_i\rangle = \frac{|e_i'\rangle}{\sqrt{\langle e_i'|e_i'\rangle}}$, $\langle e_i'|e_i'\rangle = \langle \alpha_i|\alpha_i\rangle - \sum_{j<i} \frac{|\langle \alpha_i|e_j'\rangle|^2}{\langle e_j'|e_j'\rangle}$

Ex In $\mathbb{R}^2$, basis $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}$. $|e_1'\rangle = \begin{pmatrix}1\\1\end{pmatrix}$, $|e_2'\rangle = \begin{pmatrix}2\\4\end{pmatrix} - \frac{(1\ 1)\begin{pmatrix}2\\4\end{pmatrix}}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$

$|e_1\rangle = \frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}$, $|e_2\rangle = \frac{1}{\sqrt2}\begin{pmatrix}-1\\1\end{pmatrix}$

Example Space of Polynomials over $(-1,1)$ with degree ≤ 3; basis $\{1, x, x^2, x^3\}$

$$e_1' = 1,\ e_2' = x,\ e_3' = x^2 - \frac{1}{3},\ e_4' = x^3 - \frac{3}{5}x$$

$$\Rightarrow e_1 = \frac{1}{\sqrt{2}},\ e_2 = \sqrt{\frac{3}{2}}\,x,\ e_3 = \sqrt{\frac{5}{2}} \cdot \frac{1}{2}(3x^2 - 1),\ e_4' = \sqrt{\frac{7}{2}} \cdot \frac{1}{2}(5x^3 - 3x)$$

$P_n(x)$: orthogonal Polynomials of degree n s.t. $\int_{-1}^{1} P_n^2\,dx = \frac{2}{2n+1}$

$$\Rightarrow P_n(x) = \sqrt{\frac{2}{2n+1}}\, e_{n+1}.$$

Hermite Polynomials: $w(x) = e^{-x^2}$, range: $[-\infty, \infty]$ $\int H_n^2\,dx = \sqrt{\pi}\, 2^n\, n!$

$H_0 = 1,\ H_1 = 2x,\ H_2 = 4x^2 - 2$

Can think of inner product as an operation $\langle x| : |y\rangle \to F$

Then $\langle x|$ form a vector space: $(\alpha\langle x_1| + \langle x_2|)\,|y\rangle$

$$= \langle \alpha^* x_1 + x_2 | y \rangle$$

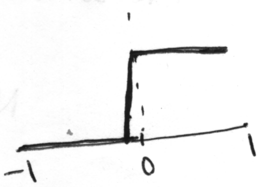
Function space

functions can be treated as vectors, eg., the polynomial fn example, or, space of all continuous fns defined on some interval (a,b).

We will use the ~~[illegible]~~ notn. $|f\rangle$ for an "abstract defn." of the fn., $f(x)$ treated as set of numbers ~~dep~~ denoting $|f\rangle$ in some basis.

Can define, eg., inner product : $\langle f|g\rangle = \int f^*(x)\, g(x)\, w(x)\, dx$

Continuous fn.s : not complete. eg. the fn.s

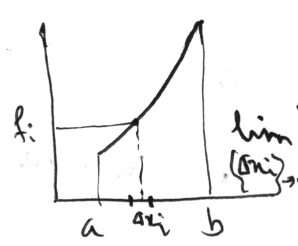


$$f_n(x) = 0 \text{ for } -1 \le x \le -\tfrac{1}{n}$$
$$= 1 \text{ for } 0 < x \le 1$$
$$= 1 + nx \text{ for } -\tfrac{1}{n} < x < 0$$

Cauchy sequence for $f_n(x)$ (under inner product norm)

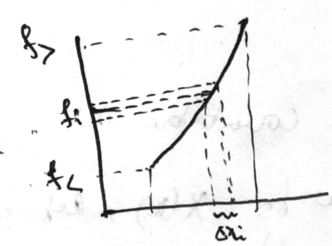
Converges to discontinuous fn.

To get to a generalisation of space of continuous fns which will be complete : first need to find a way to integrate over fns with various discontinuities.



$\lim_{\{\Delta x_i\}\to 0} \sum_i f_i \Delta x_i = \int f(x)\, dx$

Riemann



for interval δf_i around f_i
$\Rightarrow$ find Δx_i

$\lim_{\{\delta f_i\}\to 0} \sum f_i \Delta x_i = \int f(x)\, dx$

Lebesgue

More formally, on the total interval, thought of as collection of all points, we define a measure such that i) each subset $E_i \Rightarrow \mu(E_i) \geq 0$

$\mu(\phi) = 0$ for an empty set

ii) if $E_1 \cap E_2 = 0$, $\mu(E_1 + E_2) = \mu(E_1) + \mu(E_2)$

$\Rightarrow$ if $E_1 \cap E_2 = F$, $E_1 + E_2 = (\underbrace{E_1 - F}_{E_1'}) \cup (E_2 - F) \cup F$

$E_1 = E_1' + F \Rightarrow \mu(E_1') = \mu(E_1) - \mu(F)$

$\Rightarrow \mu(E_1 + E_2) = \mu(E_1) + \mu(E_2) - \mu(E_1 \cap E_2)$

To match with Riemann integral: $(a,b) \to \mu = |a-b|$

Now $(a,b) = (a,c) \cup [c] \cup (c,b) \Rightarrow \mu(c) = 0$

$\Rightarrow$ A countable set of points has measure 0.

For any set of points E in interval (a,b), let us take a set of nonoverlapping segments Λ_i, length l_i, such that $E \subset \bigcup_i \Lambda_i$. Form $\sum l_i < L = b-a$

$\mu_{ext}(E) = \min(\sum_i l_i)$

Let E': ~~of poi~~ complement of E in (a,b).

$\mu_{int}(E) = L - \mu_{ext}(E')$

$0 \leq \mu_{int}(E) \leq \mu_{ext}(E) \leq L$

if $\mu_{int}(E) = \mu_{ext}(E)$

Now for a countable set $\{x_n\}$, cover each point by a segment of length $\frac{\varepsilon}{2^n} \Rightarrow \mu_{ext} \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = 2\varepsilon \to 0$ as $\varepsilon \to 0$

$\Rightarrow \mu = 0$

Rational nos are countable set $\Rightarrow$ set of measure 0.

Ex: Dirichlet fn $X(x)$, defined over $[0,1]$: $X(x) = 1$ if x rational, 0 else

$\Rightarrow \int_0^1 X(x)\,dx = 0$. $X(x) = \lim_{n\to\infty} f_n$, $f_n = \lim_{j\to\infty} [\cos(n!\pi x)]^{2k}$

Properties of Lebesgue integral:

$$\int_A (cf+g)\,d\mu = c\int_A f\,d\mu + \int_A g\,d\mu$$

if $f \geq g$ on A, $\int_A f\,d\mu \geq \int_A g\,d\mu$

if A & B disjoint, $\int_{A\cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu$

Similar to the $\mathbb{R}^n$ examples, would like to use $\int f^* g\,d\mu$ for inner product.

Problem: for discontinuous fns $\int f^* f\,d\mu = 0 \not\Rightarrow f=0$

$\Rightarrow$ we will not distinguish betn. fns which differ only on a set of measure zero (fns' equivalence class).

Functions defined on an interval (a,b) which are square integrable in the Lebesgue sense: $\int_a^b |f(x)|^2\,dx < \infty$ (or more generally, $\int_a^b |f(x)|^2\,w(x)\,dx < \infty$)

form a vector space, $L^2(a,b)$ $[L^2_w(a,b)]$

Check: if f,g are in L^2, $\int_a^b |cf+g|^2\,dx = |c|^2\int|f|^2\,dx + \int|g|^2\,dx$
$+c^*\int f^* g\,dx + c\int f g^*\,dx < \infty$

0: $f=0$ (or its equivalent)

inner product defined through $\langle f|g\rangle = \int_a^b f^* g\,dx$

The space $L^2_w(a,b)$ is complete: if $f_1, f_2, \cdots \in L^2_w(a,b)$ form a Cauchy sequence: $\lim_{k,l\to\infty}\int_a^b |f_k(x)-f_l(x)|^2\,w(x)\,dx = 0$

Then $\exists\, f(x) \in L^2_w(a,b)$ such That
$$\lim_{k\to\infty} \int_a^b |f(x) - f_k(x)|^2\, w(x)\, dx = 0$$

(The fns f_k 'converge in The mean' to the fn $f(x)$.) [Reisz-Fischer Theorem]

We would like to use a basis and expand $|f\rangle = \sum_{i=1}^{\infty} f^i |e_i\rangle$

Work with an orthonormal set: $\langle e_i|e_j\rangle = \int_a^b e_i^* e_j\, w\, dx = \delta_{ij}$

a) form partial sums $|f_k\rangle = \sum_{i=1}^{k} f^i |e_i\rangle$ where $f^i = \int e_i^* f\, w\, dx$

Then $\big||f\rangle - |f_k\rangle\big|^2 = \left(\langle f| - \sum f^{i*}\langle e_i|\right)\left(|f\rangle - \sum f^i |e_i\rangle\right)$

$= \langle f|f\rangle - \sum_i f^{i*} f^i - \sum_i f^i f^{i*} + \sum_i f^{i*} f^i = \langle f|f\rangle - \sum_i |f^i|^2 \geq 0$

Taking $k\to\infty$: $\langle f|f\rangle \geq \sum_{i=1}^{\infty} |f^i|^2$ (Bessel's inequality)

So the series $\sum_{i=1}^{\infty} |f^i|^2$ converges $\Rightarrow \lim_{M,N\to\infty} \sum_{i=M+1}^{N} |f^i|^2 \to 0$

$\Rightarrow \lim_{M,N\to\infty} \big||f_M\rangle - |f_N\rangle\big|^2 \to 0$

$\Rightarrow |f_N\rangle$ form a Cauchy sequence $\longrightarrow |g\rangle$ in L^2_w

$\lim_{N\to\infty} |g^i - f^i_N| = \lim_{N\to\infty} \langle e_i|g - f_N\rangle \leq \lim_{N\to\infty} \|g - f_N\| = 0 \Rightarrow g^i = f^i_N$

b) If the set $\{|e_i\rangle\}$ is maximal, i.e., it is not possible to add another vector orthogonal to the set (only $|0\rangle$ orthogonal to all $\{|e_i\rangle\}$): $\quad |g\rangle = |f\rangle$.

$g^i = \lim_{N\to\infty} f^i_N = f^i \Rightarrow 0 = g^i - f^i = \langle e^i|g - f\rangle \;\; \forall i$

$\Rightarrow |g - f\rangle = |0\rangle.$

$\Rightarrow$ if $\{|e_i\rangle\}$ form an orthonormal basis, $\langle f|f\rangle = \sum_{i=1}^{\infty} |f^i|^2$ Parseval's rel.

$\Rightarrow$ Completeness of L^2 implies that given a set of orthonormal basis vectors $\{|e_i\rangle\}$, any vector $|f\rangle$ in L^2 can be expanded in the infinite sum

$$f(x) = \sum_{i=1}^{\infty} f^i e_i(x), \quad \left[f^i = \int e_i^* f(x)\, dx \right]$$

where the equality implies 'convergence in the mean'.

Theorem (Weierstrass) The functions $1, x, x^2, \cdots$ form a basis of $L^2_w(a,b)$:

a) Any continuous fn. $f(x)$ can be approximated uniformly by a suitable polynomial: given ε, there is an n s.t. $|f(x) - P_n(x)| < \varepsilon$ for any $x \in [a,b]$

b) For any $|f\rangle \in L^2_w(a,b)$, $\exists$ sequence of vectors represented by continuous fns that converges to $|f\rangle$.

Therefore the orthogonal polynomials will form an orthogonal basis.

Trigonometric basis: In $L^2[-\pi,\pi]$ the set of fns

$$\frac{1}{\sqrt{2\pi}}, \left\{ \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\}, n = 1, 2, \cdots \infty$$

form an orthonormal basis.

Therefore, in this range, $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{m=1}^{\infty} b_n \cos nx$

[in the "convergence in mean" sense]

where $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$,

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$,

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$.

Equivalently, we can use the basis $\frac{1}{\sqrt{2\pi}} e^{inx}$, $n = 0, \pm 1, \pm 2, \ldots$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}, \quad C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{inx}\, dx$$

$$C_0 = a_0, \quad C_{\pm n} = \frac{a_n \mp i b_n}{2}.$$

Follows from multiparameter generalizn. of Weierstrass Theorem:

continuous fn $f(x,y)$ can be uniformly approximated by fn

$$f_n(x,y) = \sum_{0 \le m_i, m_j \le n} a^n_{ij}\, x^{m_i} y^{m_j}$$

Taking fn $f(x,y) = r f(\theta)$, on the unit circle then

$$f_n(\theta) = \sum a^n_{ij} \cos^{m_i}\theta \sin^{m_j}\theta$$

writing $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ and rearranging,

we get the Fourier series.

Actually, for uniform convergence ~~...~~ it is required that

i) $f(x)$ is periodic : $f(-\pi) = f(\pi)$

ii) continuous

iii) ~~...~~ $\frac{df}{dx}$ is continuous except finite no. of finite discontinuities

Theorem On points of discontinuity the Fourier series converges to

$$\frac{1}{2}\left[\lim_{\varepsilon \to 0} f(\theta + \varepsilon) + \lim_{\varepsilon \to 0} f(\theta - \varepsilon)\right]$$

and at the boundary, $\theta = \pm\pi$, to $\frac{1}{2}\left[f(\pi) + f(-\pi)\right]$.