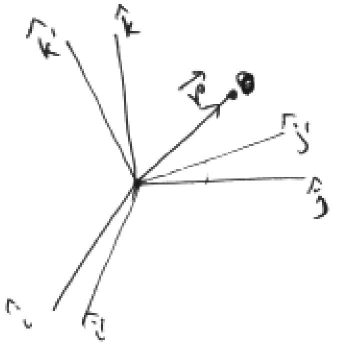
Vectors, tensors, coordinate transformations



vectors: 3-component objects that transform like coordinates under rotn.

$$\vec r = x_a \hat e_a = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad x_a = \vec r \cdot \hat e_a \qquad \hat e_a \cdot \hat e_b = \delta_{ab}$$

$\hat e_a \to \hat e'_a$ orthogonal transformn.

$$\vec r = x'_a \hat e'_a \qquad x'_a = \vec r \cdot \hat e'_a = x_b \hat e_b \cdot \hat e'_a = A_{ab} x_b$$

$$A_{ab} = \hat e'_a \cdot \hat e_b$$

$$AA^T = 1 \qquad A_{ab}A_{cb} = \sum_b \hat e'_a \cdot \hat e_b \; \hat e'_c \cdot \hat e_b = \sum_b \left(\hat e'_a \cdot \hat e_b \; \hat e_b\right) \cdot \hat e'_c = \hat e'_a \cdot \hat e'_c = \delta_{ac}$$

Example: 2-d rotn.

Arbitrary vector V: $V_a \hat e_a$ such that $V'_a = A_{ab} V_b$.

Scalar: does not transform

- Scalar product: $V_1 \cdot V_2 = V_{1a} V_{2a} \to V'_{1a} V'_{2a} = \underbrace{A_{ab} A_{ac}}_{\delta_{bc}} V_{1b} V_{2c}$

cross product: **A×B** $\Rightarrow$ introduce ϵ_{ijk} $\qquad (A \times B)_i = \epsilon_{ijk} A_j B_k$

$$\to \epsilon_{ijk} A'_j B'_k = \epsilon_{ijk} A_{jl} A_{km} A_l B_m$$

Now $\epsilon_{ijk} A_{ia} A_{jb} A_{kc} = \epsilon_{abc} \det A = \epsilon_{abc}$ for proper rotn.

$$\Rightarrow \sum_a \epsilon_{ijk} A_{ia} A_{jb} A_{kc} A_{pa} = \epsilon_{pjk} A_{jb} A_{kc} = \epsilon_{abc} A_{pa}$$

$$\Rightarrow \epsilon_{ijk} A_{jl} A_{km} = A_{ia} \epsilon_{alm} \qquad \Rightarrow (A' \times B')_i = A_{ia} (A \times B)_a$$

$$\vec\nabla = \hat e_a \frac{\partial}{\partial x_a} \qquad x'_i = A_{ij} x_j \qquad x_k = A_{ik} x'_i$$

$$\frac{\partial}{\partial x'_i} = \frac{\partial x_a}{\partial x'_i} \frac{\partial}{\partial x_a} = A_{ia} \frac{\partial}{\partial x_a}.$$

Identities $(A\times B)\cdot(C\times D) = A\cdot C\ B\cdot D - A\cdot D\ B\cdot C$

$$A\times(B\times C) = A\cdot C\ B - A\cdot B\ C$$

$$A\cdot(B\times C) = B\cdot(C\times A) = C\cdot(A\times B)$$

scalar & vector fns $\varphi'(x') = \varphi(x)$, $\vec{P}'_i(x') = A_{ij}\vec{P}_j(x)$

Now define gradient: $\vec{\nabla}\varphi = \varepsilon^a \frac{\partial}{\partial x^a}\varphi(x)$

divergence: $\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x^a} A^a(x)$

curl: $(\vec{\nabla}\times\vec{A})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}A_k(x)$

Some identities: $\nabla\cdot(\varphi A) = \nabla\varphi\cdot A + \varphi\,\nabla\cdot A$

$$\nabla\times(\varphi A) = \nabla\varphi\times A + \varphi\,\nabla\times A$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = (\nabla\times A)\cdot B - A\cdot(\nabla\times B)$$

$$\nabla(A\cdot B) = B\times(\nabla\times A) + A\times(\nabla\times B) + B\cdot\nabla A + A\cdot\nabla B$$

$$\nabla\times(A\times B) = B\cdot\nabla A - A\cdot\nabla B - B\cdot\nabla A + A\cdot\nabla B$$

$$\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$$

take line element $d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

Then $\vec{\nabla}\varphi\cdot d\vec{r} = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = d\varphi$

taking $d\vec{r}$ in surface of constant φ: $d\varphi = 0 \Rightarrow \vec{\nabla}\varphi \perp$ surface of constant φ

$\Rightarrow$ $\vec{\nabla}\varphi$: dirn of maximum space rate of change of φ.

Can also define line integrals: $\int_c \phi\, d\vec{r} = \hat{e}_a \oint_c \phi\, dx^a$

$$\int_c \vec{v}\cdot d\vec{r} = \int_c v_a\, dx_a \qquad \text{eg. } W = \int_c \vec{F}\cdot d\vec{r}$$

Surface integrals $\int \phi\, d\vec{\sigma}$, $\int \vec{v}\cdot d\vec{\sigma}$, $\int \vec{v}\times d\vec{\sigma}$ $\qquad d\vec{\sigma} = \hat{n}\, dA$

Gauss' theorem: $\displaystyle\int_S v\cdot d\sigma = \int_V \nabla\cdot v\, d\tau$ where S is a closed surface

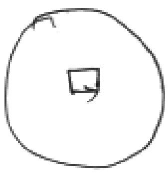
Proof: divide the volume into infinitesimal parallelepipeds.

$$-v_x\, dy\, dz + \left(v_x + \frac{\partial v_x}{\partial x} dx\right) dy\, dz = \frac{\partial v_x}{\partial x} dx\, dy\, dz$$

taking all surfaces $\Rightarrow$ on the parallelepiped

Sum over them $\Rightarrow$ Gauss' theorem.

Stokes' theorem: $\displaystyle\oint_{\partial S} \vec{v}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{v}\cdot d\vec{\sigma}$



Take S in xy-plane

Infinitesimal rectangle:

$$\text{lhs} = v_x(x,y)\,dx + v_y(x+dx, y)\,dy - v_x(x, y+dy)\,dx - v_y(x,y)\,dy$$

$$= \left(\frac{\partial}{\partial x} v_y - \frac{\partial}{\partial y} v_x\right) dx\, dy = (\nabla\times v)_z \cdot \hat{k}\, d\sigma$$

Sum over rectangles $\Rightarrow$ Stokes' theorem.

Cartesian tensors : T_{ij} s.t. $T'_{ij} = A_{ik} A_{jl} T_{kl}$

Can divide into : T_{jj} , $\frac{1}{2}(T_{ij} - T_{ji})$, $\frac{1}{2}(T_{ij} + T_{ji}) - \frac{1}{3} T_{kk}\delta_{ij}$

[properties that are frame independent : symmetry]

Cartesian tensor of rank n :

$$T_{i_1 \cdots i_n} \text{ s.t. } T'_{i_1 \cdots i_n} = A_{i_1 j_1} \cdots A_{i_n j_n} T_{j_1 \cdots j_n}$$

Pseudotensor : $T'_{i_1 \cdots i_n} = (\det A) A_{i_1 j_1} \cdots A_{i_n j_n} T_{j_1 \cdots j_n}$

invariant tensors : δ_{ij}, ε_{ijk} $\to$ pseudotensor

Tensor product : $T^1_{i_1 \cdots i_n}$, $T^2_{j_1 \cdots j_m}$

construct $T^{(3)}_{k_1 \cdots k_{n+m}} = T^1_{k_1 \cdots k_n} T^2_{k_{n+1} \cdots k_{n+m}}$ $\quad T^3 = T^1 \otimes T^2$

Show that it transforms as tensor of rank $(n+m)$

Contraction : $T'_{i_1 \cdots i_{p-1} i_{p+1} \cdots i_{q-1} i_{q+1} \cdots i_n} = T_{i_1 \cdots i_{p-1} j\, i_{p+1} \cdots i_{q-1} j\, i_{q+1} \cdots n}$

Show that it transforms as tensor of rank $n-2$

If P, Q vectors then $T_{ij} = P_i Q_j$ tensor

If $T_{ij} P_j = Q_i$, P, Q vectors, in all frames then T_{ij} is tensor.

$T'_{ij} P'_j = Q'_i \Rightarrow T'_{ij} A_{jk} P_k = A_{il} Q_l = A_{il} T_{lm} P_m$

$\Rightarrow A_{in} T'_{ij} A_{jk} P_k = T_{nm} P_m \Rightarrow (A_{in} T'_{ij} A_{jm} - T_{nm}) P_m = 0$

Taking e.g. $P_m = \delta_{mk} \Rightarrow A_{in} T'_{ij} A_{jk} = T_{nk} \Rightarrow T'_{lm} = A_{ln} A_{mk} T_{nk}$

Example 1. rigid body, angular velocity $\vec{\omega}$: angular momentum $\vec{L} = \sum_a \vec{r}_a \times m_a \vec{v}_a$

$= \sum_a \vec{r}_a \times m_a (\vec{\omega} \times \vec{r}_a)$

$\Rightarrow \vec{L} = \sum_a m_a [r_a^2 \vec{\omega} - \vec{r}_a \cdot \vec{\omega}\, \vec{r}_a] \Rightarrow L_i = \sum_a m_a [r_a^2 \delta_{ij} - r_{ai} r_{aj}] \omega_j$

$\Rightarrow L_i = I_{ij}\,\omega_j\,, \quad I_{ij} = \sum_a m_a\left[r_a^2\,\delta_{ij} - r_a^i\, r_a^j\right]$

$$\rightarrow \int dV\, \rho(\vec{r})\left(\vec{r}^{\,2}\delta_{ij} - r_i\, r_j\right)$$

2. Ohm's law $\quad \vec{j} = \sigma \vec{E} \qquad$ if $\vec{j}$ and $\vec{E}$ not in same dirn.

$$\Rightarrow j_i = \sigma_{ij} E_j \qquad \sigma_{ij} : \text{conductance tensor}$$

e.g. in presence of magnetic field.

Vectors & tensors in Minkowski space

Velocity of light constant in all frames

$\Rightarrow$ contradiction with Galilean invariance

Einstein: a) assume ~~[illegible]~~ frame invariance

b) velocity of light same

$\Rightarrow dx^2 + dy^2 + dz^2 - c^2dt^2$ should be invariant under coordinate transformn.

Think of a (3+1) dim. coordinate space (Minkowski)

(x, y, z, ct)

$$-d\tau^2 = dx^2 + dy^2 + dz^2 - dt^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \qquad g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$

coordinate transformn. $\Rightarrow \quad g_{\mu\nu}\, dx'^\mu\, dx'^\nu = g_{\mu\nu}\, \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta\, dx^\alpha dx^\beta$

$$= g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

$\Rightarrow$ Lorentz transformns Λ: $\quad g_{\mu\nu}\, \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}$

$\Rightarrow (\det \Lambda)^2 \cdot \det g = \det g \Rightarrow \det\Lambda = \pm 1$

Also put $\alpha, \beta = 0 \Rightarrow -1 = -\Lambda^0{}_0{}^2 + \sum_i \Lambda^i{}_0{}^2 \Rightarrow \Lambda^0{}_0{}^2 = 1 + \sum_i \Lambda^i{}_0{}^2 \geq 1$

If $\Lambda^0{}_0 \geq 1$, $\det\Lambda = +1 \Rightarrow$ proper Lorentz transformn.

$\Lambda^0{}_0 = 1$, $\Lambda^i{}_0 = 0 \Rightarrow$ transformns that keep dt unchanged

$\Rightarrow$ transformns that keep $d\vec{x}^2$ unchanged $\Rightarrow$ rotns

Boost: One observer O sees particle at rest

O' : particle moving with velocity v

O: $d\vec{x} = 0$, dt $\quad O'$: $dx'^i = \Lambda^i{}_0 \, dt$ $\Rightarrow \frac{dx'^i}{dt'} = v^i \Rightarrow \Lambda^i{}_0 = v^i \Lambda^0{}_0$

$dt' = \Lambda^0{}_0 \, dt$

$$\Lambda^{0\,2}_0 = 1 + \sum_i \Lambda^{i\,2}_0 \Rightarrow (1 - v^2)\Lambda^{0\,2}_0 = 1 \Rightarrow \Lambda^0{}_0 = \gamma \quad \Lambda^i{}_0 = v_i \gamma$$

One choice satisfying $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \, g_{\mu\nu} = g_{\alpha\beta}$: $\Lambda^i{}_j = \delta_{ij} + v_i v_j \frac{\gamma - 1}{\vec{v}^2}$

$\Lambda^0{}_j = \gamma v_j$

Taking velocity in x direcn. : $\Lambda^\mu{}_\alpha = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ a & b & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\Lambda g \Lambda^T = g \Rightarrow b = \gamma, \; a = v\gamma$$

Time dilatation: one observer sees clock at rest, two ticks betn. dt

$d\tau^2 = dt^2$

second observer: clock moving with vel v, ticks betn. dt', clock moved $v\,dt'$

$$\Rightarrow d\tau^2 = dt'^2 - v^2 dt'^2 = (1 - v^2) dt'^2 \Rightarrow dt' = \gamma dt$$

<u>Proof that $d\tau^2$ invariance leads to Lorentz eqn.</u>:

$$d\tau'^2 = -g_{\mu\nu} \, dx'^\mu dx'^\nu = -g_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} dx^\alpha dx^\beta$$

$$\Rightarrow g_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} = g_{\alpha\beta}$$

$$\frac{\partial}{\partial x^\gamma} \Rightarrow g_{\mu\nu} \left[\frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\gamma} \frac{\partial x'^\nu}{\partial x^\beta} + \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial^2 x'^\nu}{\partial x^\beta \partial x^\gamma} \right] = 0$$

add to this same eqn. with α and γ exchanged, and subtract eqn. with γ & β exchanged

$\Rightarrow 0 = g_{\mu\nu} \frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\gamma} \frac{\partial x'^\nu}{\partial x^\beta}$ Since $g_{\mu\nu}$, $\frac{\partial x'^\nu}{\partial x^\beta}$ are nonsingular $\Rightarrow \frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\gamma} = 0$.

(Contravariant) four-vector: V^μ transforms like dx^μ : $V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu = \Lambda^\mu{}_\nu V^\nu$

note $\sum_\mu V^\mu V^\mu$ not a scalar $\rightarrow$ transforms to $\sum_\mu \underbrace{\Lambda^\mu{}_\nu \Lambda^\mu{}_\lambda}_{\neq \delta_{\nu\lambda}} V^\nu V^\lambda$: would have made $\sum_\mu dx^\mu dx^\mu$ invariant

$g_{\mu\nu} dx^\nu \rightarrow g_{\mu\nu} \frac{\partial x'^\nu}{\partial x^\lambda} dx^\lambda = g_{\mu\nu} \Lambda^\nu{}_\lambda dx^\lambda$ Also $g_{\mu\nu} \Lambda^\nu{}_\lambda \Lambda^\mu{}_\sigma = g_{\lambda\sigma}$

define $\Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}$: $\Lambda_\mu{}^\nu \Lambda^\mu{}_\lambda = \delta^\nu_\lambda$, $\Lambda_\mu{}^\nu \Lambda^\lambda{}_\nu = \delta^\lambda_\mu$

$\Rightarrow g_{\mu\nu} \Lambda^\nu{}_\lambda = \Lambda_\mu{}^\sigma g_{\sigma\lambda} \Rightarrow g_{\mu\nu} dx^\nu \rightarrow \Lambda_\mu{}^\sigma g_{\sigma\lambda} dx^\lambda$

Covariant vector: V_μ transforming like $g_{\mu\nu} dx^\nu$: $V'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu$

'lowering' & 'raising' of indices: $V_\mu = g_{\mu\nu} V^\nu$, $V^\mu = g^{\mu\nu} V_\nu$

where $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda$ $\quad g^{\mu\nu} = (-1, 1, 1, 1)$

$\frac{\partial}{\partial x^\alpha} \rightarrow \frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta}$ gradient: covariant vector

tensor: $T^\mu{}_{\nu\rho} \rightarrow T'^\mu{}_{\nu\rho} = \Lambda^\mu{}_\alpha \Lambda_\nu{}^\beta \Lambda_\rho{}^\gamma \; T^\alpha{}_{\beta\gamma}$

Direct product: $T^\alpha{}_\beta{}^\gamma = A^\alpha{}_\beta B^\gamma$

contraction: $T^{\alpha\gamma} = T^\alpha{}_\mu{}^{\mu\gamma} = g_{\mu\nu} T^{\alpha\mu\gamma\nu}$.

Derivative: $T_\alpha{}^{\beta\gamma} = \frac{\partial}{\partial x^\alpha} T^{\beta\gamma}$ is a tensor.

$g_{\mu\nu}$: invariant tensor So are $g^{\mu\nu}$, $\delta^\mu{}_\nu$

define $\varepsilon^{\mu\nu\rho\sigma}$: completely antisymmetric, $\varepsilon^{0123} = 1$

$$\varepsilon^{\mu\nu\rho\sigma} \to \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta\, \varepsilon^{\alpha\beta\gamma\delta} = \varepsilon^{\mu\nu\rho\sigma} \det \Lambda$$

$$\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon^{\mu\nu\rho\sigma}$$

Generalizn. of Newton's eqn. $m\frac{d^2\vec{x}}{dt^2} = \vec{F}$: $\quad m\frac{d^2x^\alpha}{d\tau^2} = f^\alpha$

going to frame where particle is at rest, $m\frac{d^2\vec{x}}{dt^2} = \vec{F}$

so in this frame $f^0 = 0$, $\vec{f} = \vec{F}$

in other frame $f'^\alpha = \Lambda^\alpha{}_\beta f^\beta$

[note that initial conditions $x^\mu(\tau)$ should satisfy $-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 1$.

Consistency: if it is satisfied at $\tau = \tau_0$, it will be satisfied at all times if $\frac{d}{d\tau}(\text{lhs}) = -2g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\frac{dx^\nu}{d\tau} = 0$ or $g_{\mu\nu} f^\mu \frac{dx^\nu}{d\tau} = 0$

This is true since it is true in the rest frame.]

Maxwell's eqns $\quad \nabla\cdot E = \rho$, $\nabla\times B = \frac{\partial E}{\partial t} + J$, $\nabla\cdot B = 0$, $\nabla\times E = -\frac{1}{c^2}\frac{\partial B}{\partial t}$

$$E_i = \partial_0 A^i - \partial_i A^0, \quad B_i = \varepsilon_{ijk}\partial_j A_k$$

$\Rightarrow \quad E_i = F^{0i}$, $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$B_1 = F^{23}$ etc.

$$\partial_\alpha F^{\alpha\beta} = -j^\beta, \quad J = (\rho, \vec{J})$$

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0$$

for a charged particle in EM field : $f^\alpha = eF^{\alpha\beta} v_\beta \qquad v^\mu = \frac{dx^\mu}{d\tau}$

General coordinate transformation: $x^\mu \to x'^\mu(x^\mu)$

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu ,$$

Contravariant vector: $V'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} V^\nu$

Covariant vector: $V'_\mu = \dfrac{\partial x^\nu}{\partial x'^\mu} V_\nu$

invariant length element ('proper time'): $g_{\mu\nu}\, dx^\mu dx^\nu$

$$g'_{\mu\nu}\, dx'^\mu dx'^\nu = g'_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} dx^\alpha dx^\beta$$

$$\Rightarrow g'_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} = g_{\alpha\beta} \quad \text{or} \quad g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} .$$

Tensor $T^\mu{}_{\nu\lambda} \to T'^\mu{}_{\nu\lambda} = \dfrac{\partial x'^\mu}{\partial x^\alpha} \dfrac{\partial x^\beta}{\partial x'^\nu} \dfrac{\partial x^\gamma}{\partial x'^\lambda} T^\alpha{}_{\beta\gamma}$.

$$\delta^\mu{}_\nu \to \delta'^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \delta^\alpha{}_\beta = \delta^\mu{}_\nu .$$

define $g^{\mu\nu}$: $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu{}_\lambda$

$$\Rightarrow g'^{\mu\nu} g'_{\nu\lambda} = \delta^\mu{}_\lambda = g'^{\mu\nu} \cdot \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

$$\Rightarrow g'^{\mu\nu} \cdot \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} = g^{\alpha\beta} \Rightarrow g'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} g^{\alpha\beta}$$

tensor algebra: if $A^\mu{}_\nu$, $B^\mu{}_\nu$ tensors, so is $T^\mu{}_\nu = A^\mu{}_\nu + B^\mu{}_\nu$

$T^\mu{}_\nu{}^{\rho\sigma} = A^\mu{}_\nu B^{\rho\sigma}$ tensor (direct product)

contraction: $T^{\mu\rho} = T^\mu{}_\nu{}^{\rho\nu}$ etc

Raising & lowering of indices: $T^{\mu\nu}{}_\rho = g^{\nu\lambda} T^\mu{}_{\lambda\rho}$ etc.

Derivatives: if φ is scalar, $\frac{\partial \varphi}{\partial x^\mu}$ covariant tensor. Gradient.

$$\frac{\partial V^\mu}{\partial x^\lambda} : \text{tensor } T^\mu{}_\lambda \,?$$

$$\frac{\partial V^\mu}{\partial x^\lambda} \to \frac{\partial V'^\mu}{\partial x'^\lambda} = \frac{\partial x^\beta}{\partial x'^\lambda} \frac{\partial}{\partial x^\beta}\left(\frac{\partial x'^\mu}{\partial x^\nu} V^\nu\right)$$

$$= \frac{\partial x^\beta}{\partial x'^\lambda} \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial V^\nu}{\partial x^\beta} + \frac{\partial x^\beta}{\partial x'^\lambda} \frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\nu} V^\nu$$

To form a tensor $T^\mu{}_\lambda = \frac{\partial V^\mu}{\partial x^\lambda} + \underbrace{\Gamma^\mu{}_{\lambda\kappa}}_{\text{affine connection}} V^\kappa$:

$$\Gamma^\mu{}_{\lambda\kappa} V^\kappa \to \Gamma'^\mu{}_{\lambda\kappa} V'^\kappa = \frac{\partial x^\beta}{\partial x'^\lambda} \frac{\partial x'^\mu}{\partial x^\nu} \Gamma^\nu{}_{\beta\sigma} V^\sigma - \frac{\partial x^\beta}{\partial x'^\lambda} \frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\nu} V^\nu$$

$$= \Gamma'^\mu{}_{\lambda\kappa} \frac{\partial x'^\kappa}{\partial x^\sigma} V^\sigma$$

$$\Rightarrow \Gamma'^\mu{}_{\lambda\kappa} = \frac{\partial x^\beta}{\partial x'^\lambda} \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\sigma}{\partial x'^\kappa} \Gamma^\nu{}_{\beta\sigma} - \frac{\partial x^\beta}{\partial x'^\lambda} \frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\kappa}$$

$$= \frac{\partial x^\beta}{\partial x'^\lambda} \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\sigma}{\partial x'^\kappa} \Gamma^\nu{}_{\beta\sigma} - \frac{\partial x^\sigma}{\partial x'^\kappa} \frac{\partial^2 x'^\mu}{\partial x'^\lambda \partial x^\sigma}$$

$$\frac{\partial}{\partial x'^\lambda}\left[\frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\kappa} = \delta^\mu_\kappa\right] = 0 = \frac{\partial x^\sigma}{\partial x'^\kappa} \frac{\partial^2 x'^\mu}{\partial x'^\lambda \partial x^\sigma} + \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x'^\lambda \partial x'^\kappa}$$

$$\Rightarrow \Gamma'^\mu{}_{\lambda\kappa} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\beta}{\partial x'^\lambda} \frac{\partial x^\sigma}{\partial x'^\kappa} \Gamma^\nu{}_{\beta\sigma} + \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x'^\lambda \partial x'^\kappa}$$

To show that $\Gamma^\mu{}_{\lambda\kappa} = \left\{ {\mu \atop \lambda\kappa} \right\} = \frac{1}{2} g^{\mu\nu}\left[\frac{\partial g_{\nu\kappa}}{\partial x^\lambda} + \frac{\partial g_{\nu\lambda}}{\partial x^\kappa} - \frac{\partial g_{\lambda\kappa}}{\partial x^\nu}\right]$

$$\frac{\partial g_{\nu\kappa}}{\partial x^\lambda} \to \frac{\partial}{\partial x'^\lambda}\left(\frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\kappa} g_{\alpha\beta}\right) = \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\kappa} \frac{\partial g_{\alpha\beta}}{\partial x^\tau} \frac{\partial x^\tau}{\partial x'^\lambda} + g_{\alpha\beta}\left[\frac{\partial^2 x^\alpha}{\partial x'^\lambda \partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\kappa} + \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial^2 x^\beta}{\partial x'^\lambda \partial x'^\kappa}\right]$$

$$\Rightarrow [\nu\kappa\lambda] \to \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\kappa} \frac{\partial x^\tau}{\partial x'^\lambda} [\alpha\beta\tau] + g_{\alpha\beta} \cdot 2 \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial^2 x^\beta}{\partial x'^\lambda \partial x'^\kappa}$$

$$\Rightarrow \left\{ {\mu \atop \lambda\kappa} \right\} \to \left\{ {\mu' \atop \lambda'\kappa'} \right\} = \frac{1}{2} g^{\mu'\nu'} [\nu'\kappa'\lambda'] = \frac{1}{2} \cdot \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\delta} g^{\rho\sigma} \cdot \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\kappa} \frac{\partial x^\tau}{\partial x'^\lambda} [\alpha\beta\tau]$$

$$+ g^{\rho\sigma} \cdot \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\delta} g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial^2 x^\beta}{\partial x'^\lambda \partial x'^\kappa}$$

$$= \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\beta}{\partial x'^\kappa} \frac{\partial x^\tau}{\partial x'^\lambda} \cdot \frac{1}{2} g^{\rho\alpha} [\alpha\beta\tau] + \frac{\partial x'^\mu}{\partial x^\beta} \frac{\partial^2 x^\beta}{\partial x'^\lambda \partial x'^\kappa}$$

$\Rightarrow \quad \Gamma^\mu_{\lambda\kappa} - \left\{ {\mu \atop \lambda\kappa} \right\}$ transforms like a tensor; is zero in local inertial frame

$\Rightarrow \quad \Gamma^\mu_{\lambda\kappa} = \left\{ {\mu \atop \lambda\kappa} \right\}$.

Similarly, covariant derivative for V_μ:

$$\frac{\partial V'_\mu}{\partial x'^\nu} = \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial}{\partial x^\rho}\left[\frac{\partial x^\alpha}{\partial x'^\mu} V_\alpha\right] = \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial V_\alpha}{\partial x^\rho} + \frac{\partial^2 x^\alpha}{\partial x'^\nu \partial x'^\mu} V_\alpha$$

$$\Gamma'^\lambda_{\mu\nu} V'_\lambda = \left[\frac{\partial x'^\lambda}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\nu}\Gamma^\alpha_{\beta\gamma} + \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}\right]\frac{\partial x^\delta}{\partial x'^\lambda} V_\delta$$

$$= \frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\nu}\Gamma^\alpha_{\beta\gamma} V_\alpha + \frac{\partial^2 x^\delta}{\partial x'^\mu \partial x'^\nu} V_\delta$$

$\Rightarrow \quad V_{\mu;\nu} = \dfrac{\partial V_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} V_\lambda$ is a tensor.

More generally, $T^{\mu\nu}{}_{\lambda;\rho} = \dfrac{\partial}{\partial x^\rho} T^{\mu\nu}{}_\lambda + \Gamma^\mu_{\rho\kappa} T^{\kappa\nu}{}_\lambda + \Gamma^\nu_{\rho\kappa} T^{\mu\kappa}{}_\lambda - \Gamma^\kappa_{\lambda\rho} T^{\mu\nu}{}_\kappa$

Note: $V_{\mu;\nu} - V_{\nu;\mu} = \dfrac{\partial V_\mu}{\partial x^\nu} - \dfrac{\partial V_\nu}{\partial x^\mu}$ Also $g_{\mu\nu;\lambda} = 0 = g^{\mu\nu}{}_{;\lambda} = \delta^\mu_{\nu;\lambda}$

Simple examples in 2-d:

① r-θ: $x = r\cos\theta$, $y = r\sin\theta$ $r = \sqrt{x^2+y^2}$, $\tan\theta = \dfrac{y}{x}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial\theta}{\partial x} = -\frac{y}{r^2}, \quad \frac{\partial\theta}{\partial y} = \frac{x}{r^2}$$

$$\frac{\partial x}{\partial r} = \frac{x}{r}, \quad \frac{\partial y}{\partial r} = \frac{y}{r}, \quad \frac{\partial x}{\partial\theta} = -y, \quad \frac{\partial y}{\partial\theta} = x$$

$ds^2 = dr^2 + r^2 d\theta^2 \qquad g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$

$$V^r = \frac{x}{r}V^x + \frac{y}{r}V^y, \quad V^\theta = -\frac{y}{r^2}V^x + \frac{x}{r^2}V^y$$

$$V^x = \frac{x}{r}V^r - yV^\theta, \quad V^y = \frac{y}{r}V^r + xV^\theta$$

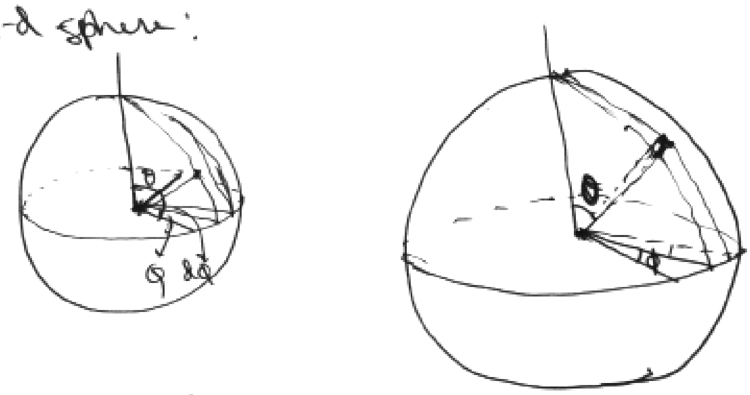
$g_{\mu\nu}V^\mu V^\nu = 2KE$ for unit mass

$V^x_\theta = -y\omega$
$V^y_\theta = x\omega$

$$\Gamma^{\mu}_{\nu\alpha} = \frac{1}{2} g^{\mu\rho}\left[\frac{\partial g_{\rho\nu}}{\partial x^{\alpha}} + \frac{\partial g_{\rho\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\nu\alpha}}{\partial x^{\rho}}\right]$$

$$\Gamma^{r}_{rr} = 0 = \Gamma^{\theta}_{rr}, \quad \underline{\Gamma^{r}_{\theta\theta} = -r}, \quad \Gamma^{r}_{r\theta} = 0, \quad \underline{\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}}, \quad \Gamma^{\theta}_{\theta\theta} = 0.$$

(2) On 2-d sphere:



$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2$$

$$g_{\mu\nu} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

Nonzero connections are: $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$, $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$

(4)

(3) $g_{\mu\nu} = (-1, 1, r^2, r^2\sin^2\theta) \Rightarrow$ Minkowski space

(4) $g_{\mu\nu} = \left(-\left(1 - \frac{2MG}{r}\right), \left(1-\frac{2MG}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right)$ static isotropic metric in presence of gravitational field (Schwarzschil

$\frac{dx^\mu}{d\tau}$: vector

$$\frac{d^2x'^\mu}{d\tau^2} \Rightarrow \frac{d}{d\tau}\left(\frac{dx'^\mu}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{\partial x'^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right) = \frac{\partial x'^\mu}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}$$

$$\frac{d^2x'^\mu}{d\tau^2} + \Gamma'^{\mu}_{\nu\lambda}\frac{dx'^\nu}{d\tau}\frac{dx'^\lambda}{d\tau} \Rightarrow \frac{\partial x'^\mu}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2 x'^\mu}{\partial x^\lambda\partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}$$

$$+ \frac{\partial x'^\mu}{\partial x^\lambda}\Gamma^{\lambda}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} + \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\nu\partial x'^\lambda}\cdot\frac{\partial x'^\nu}{\partial x^\rho}\frac{\partial x'^\lambda}{\partial x^\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}$$

$$= \frac{\partial x'^\mu}{\partial x^\nu}\left(\frac{d^2x^\nu}{d\tau^2} + \Gamma^{\nu}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)$$

tensor density of weight w : object that transforms like a tensor $\times \left(\det\left(\frac{\partial x'}{\partial x}\right)\right)^w$

e.g. $g = -\det g_{\mu\nu}$ $\qquad g_{\mu\nu} \to \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \Rightarrow g' = \left(\det\left(\frac{\partial x}{\partial x'}\right)\right)^2 \cdot g$

$$\Rightarrow g : \ w = -2$$

d^4x : $w = 1$ $\qquad d^4x\sqrt{g}$: invariant volume element

tensor density of rank $w \equiv g^{-w/2}\cdot$ tensor

$\varepsilon^{\mu\nu\rho\sigma}$: antisymmetric, $\varepsilon^{0123} = 1$ [in any coordinate system]

$$: \quad \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}\frac{\partial x'^\gamma}{\partial x^\rho}\frac{\partial x'^\sigma}{\partial x^\sigma}\varepsilon^{\mu\nu\rho\sigma} = \det\left(\frac{\partial x'}{\partial x}\right)\cdot\varepsilon^{\alpha\beta\gamma\sigma}$$

$\Rightarrow g^{-1/2}\varepsilon^{\mu\nu\rho\sigma}$ is a tensor.

divergence : $V^\mu{}_{;\mu} = \frac{\partial V^\mu}{\partial x^\mu} + \Gamma^\mu_{\mu\lambda}V^\lambda$

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2}g^{\mu\rho}\frac{\partial g_{\rho\mu}}{\partial x^\lambda} = \frac{1}{2}\mathrm{Tr}\, M^{-1}\frac{\partial}{\partial x^\lambda}M, \quad M = \text{matrix } g_{\mu\rho}$$

$$= \frac{1}{2}\frac{\partial}{\partial x^\lambda}\ln\det M,$$

using $\delta\ln\det M = \ln\det(M+\delta M) - \ln\det M$

$$= \ln\det(1+M^{-1}\delta M) \Rightarrow \ln(1+\mathrm{Tr}\, M^{-1}\delta M) = \mathrm{Tr}\, M^{-1}\delta M$$

$$\Rightarrow \Gamma^\mu_{\mu\lambda} = \frac{\partial}{\partial x^\lambda}\ln\sqrt{g} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\lambda}\sqrt{g} \Rightarrow V^\mu{}_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{g}\,V^\mu\right)$$

$\Rightarrow \int d^4x\sqrt{g}\,V^\mu{}_{;\mu} = 0$ if V^μ vanishes at infinity.

$$T^{\mu\nu}{}_{;\mu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\mu\lambda}T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda}T^{\mu\lambda}$$

if $T^{\mu\nu}$ antisymmetric: $T^{\mu\nu}{}_{;\mu} = \frac{\partial T^{\mu\nu}}{\partial x^\mu} + \frac{1}{\sqrt{g}}\left(\partial_\lambda\sqrt{g}\right)T^{\lambda\nu} = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\,T^{\mu\nu}\right)$

Maxwell's eqns in local inertial frame: $\partial_\mu F^{\mu\nu} = -j^\nu$

in general frame: $D_\mu F^{\mu\nu} = -j^\nu \Rightarrow \frac{1}{\sqrt g}\,\partial_\mu\left(\sqrt g\, F^{\mu\nu}\right) = -j^\nu$

$\Rightarrow \partial_\mu\left(\sqrt g\, F^{\mu\nu}\right) = -\sqrt g\, j^\nu$

$\tilde F^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \Rightarrow g^{-1/2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

$D_\mu \tilde F^{\mu\nu} = 0 \Rightarrow \frac{1}{\sqrt g}\,\partial_\mu\left[\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}\right] = 0 \Rightarrow \partial_\mu\left[\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}\right] = 0.$

Lorentz force: $f^\alpha = e\, F^\alpha{}_\beta\, v^\beta = e\, g_{\beta\gamma} F^{\alpha\gamma} v^\beta.$

"Covariant curl": $V_{\mu;\nu} - V_{\nu;\mu} = \frac{\partial V_\mu}{\partial x^\nu} - \frac{\partial V_\nu}{\partial x^\mu}$ using symmetry of $\Gamma^\lambda{}_{\mu\nu}$

<u>Orthogonal curvilinear coordinate systems:</u> $g_{ij} = h_i^2\,\delta_{ij}$, $i,j = 1,2,3$

[no sums in this section] $g^{ij} = h_i^{-2}\,\delta_{ij}$

$V_i = h_i^2\, V^i$ define $\bar V_i = \bar V^i = h_i V^i = h_i^{-1} V_i$ [such that $P\cdot Q = \sum_i \bar P_i \bar Q_i$]

gradient: $\nabla_i S = \overline{S_{;i}} = \frac{1}{h_i}\frac{\partial S}{\partial x^i}$ [no sum]

divergence: $\nabla\cdot V = \frac{1}{\sqrt g}\sum_i \partial_i\left[\sqrt g\, V^i\right] = \frac{1}{h_1h_2h_3}\sum_i \partial_i\left[\frac{h_1h_2h_3}{h_i}\cdot \bar V^i\right]$

$\sum_{jk} g^{-1/2}\varepsilon^{ijk} V_{k;j}$ contravariant vector $\Rightarrow$ so curl can be written

$(\nabla\times V)_i = h_i \sum_{jk} g^{-1/2}\varepsilon_{ijk}\,\partial_j\left(h_k \bar V_k\right)$

Laplacian: $\nabla^2\varphi = \frac{\partial}{\partial x^i}\left(\frac{\partial\varphi}{\partial x^i}\right) \Rightarrow D_i\left[g^{ij}\frac{\partial\varphi}{\partial x^j}\right]$

$= \frac{1}{\sqrt g}\sum_i \frac{\partial}{\partial x^i}\left[\frac{\sqrt g}{h_i^2}\frac{\partial\varphi}{\partial x^i}\right].$

Example: (r,θ,φ): $\sqrt g = r^2\sin\theta$, $h_r = 1$, $h_\theta = r^2$, $h_\varphi = r^2\sin^2\theta$

$\Rightarrow \nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\varphi}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\varphi}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\varphi^2}.$