Assignment 2

Due date: 26th March

1. Define $\rho(x-y) = \langle 0| \, \frac{1}{2} \, [\phi(x), \phi(y)] \, |0\rangle$ for the real scalar field. Its Fourier transform, $\rho(p_0, \vec{p}) = \int d^4x \; \rho(x) \; \exp(ip.x)$, is called the spectral function.

   (a) Show that $\rho(p_0, \vec{p}) \; = \; \text{Im} \; iD_R(p_0, \vec{p})$, where $D_R(p_0, \vec{p})$ is the Fourier transform of the retarded Green's function.

   (b) Show that for $p_0 > 0$, $\rho(p_0, \vec{p}) \; = \; \text{Im} \; iD_F(p_0, \vec{p})$, where $D_F(p_0, \vec{p})$ is the Feynman propagator in momentum space.

   (c) Prove the sum rule

   $$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \; 2p_0 \; \rho(p_0, \vec{p}) \; = \; 1.$$

   (d) Calculate $\rho(p_0, \vec{p})$ for the free scalar field.

2. Use the time evolution operator, $U(t_1, t_2)$, defined in the class to show

   (a)

   $$U(t_1, t_2) = e^{iH_0 t_1} \; e^{-iH(t_1 - t_2)} \; e^{-iH_0 t_2},$$

   where H_0, H are the free and the full Hamiltonian, respectively.

   (b)

   $$U(t_1, t_2) \; U(t_2, t_3) \; = \; U(t_1, t_3), \qquad\qquad t_1 > t_2 > t_3.$$

   (c) the unitarity of the S matrix: $S^\dagger S \; = \; SS^\dagger \; = \; 1$.

3. Take the theory of two scalar fields Φ, ϕ, with masses M, m and the interaction term $\mathcal{L} \; = \; \mu \Phi \phi \phi$. Calculate the scattering cross-section $\phi(k_1)\phi(k_2) \; \rightarrow \; \phi(p_1)\phi(p_2)$ to lowest order.