Assignment 4

Due date: 5th May

1. In the first assignment, you had obtained the conserved charge densities for Lorentz transformation,

$$M^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu .$$

Of course, the corresponding quantum mechanical operators are

$$M^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu).$$

a) Check that $J_i = \frac{1}{2}\epsilon_{0ijk} M^{jk}$ are the rotation generators, and $K_i = -M_{0i}$ are the Boost generators.

b) Form the operators $A_i = (J_i + iK_i)/2,\ B_i = (J_i - iK_i)/2$. Show that $[A_i, A_j] = i\epsilon_{ijk}A_k,\ [B_i, B_j] = i\epsilon_{ijk}B_k,\ [A_i, B_j] = 0$.

c) Show that $[J_i, P_j] = i\epsilon_{ijk}P_k,\ [K_i, P_j] = -iP^0\delta_{ij}$.

d) Form the operator $W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma$. Show that $W^\mu W_\mu$ and $P^\mu P_\mu$ commute with J_i, K_i, P_μ and W_μ.

2. Define the "vertex functions"

$$\Gamma_n(x_1, x_2, ..., x_n) \;=\; \frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1)\delta\phi(x_2)...\delta\phi(x_n)}$$

where $\Gamma[\phi] = \int dx J(x)\phi(x) - W[J],\quad \phi(x) = \delta W[J]/\delta J(x)$ and $W[J]$ is the generator of the connected Green's functions. Express Γ_{2-4} in terms of the Connected Green's functions G^c_{2-4}.

3. The 4-point vertex function to $O(\lambda^2)$, in $\overline{\mathrm{MS}}$ scheme, was evaluated in the class:

$$\Gamma_4(p_1, p_2, p_3, p_4) \;=\; -i\left[\lambda + \frac{\lambda^2}{32\pi^2}\int_0^1 dx\, \log\frac{m^2 - i\epsilon - sx(1-x)}{\mu^2} \;+ (s\to t)\; + (s \to u)\right].$$

Show that, for $s > 4m^2$,

$$-2\,\mathrm{Im}\, i\Gamma_4(p_1, p_2, p_3, p_4) \;=\; \frac{\lambda^2}{2}\int \frac{d^3k_1}{(2\pi)^3 2E_1}\int \frac{d^3k_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^4(p_1+p_2-k_1-k_2).$$

4. Take the theory of two scalar fields Φ, ϕ, with masses M, m and the interaction term $\mathcal{L} = \mu\Phi\phi\phi$. (Assignment 2, Problem 3.)

   a) Work out the Feynman rules for renormalized perturbation theory, and calculate the counterterms (in $\overline{\text{MS}}$ scheme) to one-loop order. Will you need any extra type of counterterm at higher loops?

   b) You had calculated the scattering cross-section $\phi(k_1)\phi(k_2) \rightarrow \phi(p_1)\phi(p_2)$ to lowest (nontrivial) order. Write the expression for the cross-section correct to the next (nontrivial) order. (You do not need to evaluate all the finite integrals, it will suffice to give the results in the integral form.)