Advanced Quantum Mechanics

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Lecture #26

QM of Relativistic Particles
Inertial Frames and Newton's Laws.

Two observers travelling with a velocity \( v \) wrt each other should write the same equation of motion for a given system.

Let the co-ordinates used by the 2 observers be

\[
\begin{align*}
A & \longrightarrow (x,y,z,t) \\
B & \longrightarrow (x',y',z',t')
\end{align*}
\]

Galilean Transformation

\[
\begin{align*}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{align*}
\]

The co-ordinates are not invariant in the two frames.

Newton's Law:

\[
m\ddot{r} = \vec{F}(\vec{r}, t) \rightarrow m\ddot{\vec{r}} = \vec{F}'(\vec{r}', t')
\]

If Newton's law is to look the same, \( F'(r', t') = F(r, t) \) at all points connected by Galilean Transformation. Note that the fn \( F \) can change to another fn \( F' \).

Equivalent statement: The force is same in all inertial frames.
Inertial Frames and Lagrangian

Two observers travelling with a velocity \( v \) wrt each other should write the same equation of motion for a given system

Let the co-ordinates used by the 2 observers be

\[
A \rightarrow (x,y,z,t) \quad B \rightarrow (x',y',z',t')
\]

Galilean Transformation

\[
x' = x - vt \quad y' = y \quad z' = z \quad t' = t
\]

Lagrangian:

\[
\mathcal{L} = \frac{m}{2} \ddot{r}^2 - V(\vec{r})
\]

\[
= \frac{m}{2}([\dot{\vec{r}}]^2 + 2\dot{\vec{r}} \cdot \vec{v} + \vec{v}^2) - V(\vec{r} + \dot{\vec{r}}t)
\]

Note that as long as \( v \) is constant in time, the Euler Lagrange Equations look the same in both frames, as long as \( V(r)=V'(r'+vt') \) for all points connected by Galilean transformations.

Once again note that functional form of both \( L \) and \( V \) changes, but eqn. of motion looks the same

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0 \rightarrow \frac{d}{dt'} \frac{\partial \mathcal{L}'}{\partial \dot{x}'} - \frac{\partial \mathcal{L}'}{\partial x'} = 0
\]
Constant Speed of light and Lorentz Transformation

Let the co-ordinates used by the 2 observers be

$$ A \rightarrow (x,y,z,t) \quad B \rightarrow (x',y',z',t') $$

Galilean Transformation

$$ x' = x - vt \quad y' = y \quad z' = z \quad t' = t $$

Galilean Transformations are incompatible with the observed phenomenon that speed of light in vacuum is same in all inertial frames and is equal to $c = 3 \times 10^{10} \text{ m/s}$

Lorentz Transformations

$$ x' = \gamma (x - vt) \quad y' = y \quad z' = z \quad ct' = \gamma (ct - \beta x) $$

$$ \beta = \frac{v}{c} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} $$

Check that: If $\frac{dx}{dt} = c$ in one frame, $\frac{dx'}{dt'} = c$ in all frames connected by LT

Use $c = 1$ to set units $\hbar = 1, c = 1 \quad E = M = T^{-1} = L^{-1}$

Lorentz Boost

$$ \begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\eta \\ -\eta & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} $$

Rapidity

$$ \eta = \beta \gamma $$

$$ \eta^2 - \eta^2 = 1 $$

$$ t^2 - x^2 - y^2 - z^2 \text{ is invariant under LT} $$
Lorentz Transformations

Generalize: Set of Linear transformations of (t,x,y,z) which keeps $s^2=t^2-x^2-y^2-z^2$ invariant

The Lorentz transformations form a continuous non-compact group $O(3,1)$

Four Vector: $x^\mu = (x^0, x^i) = (t, \vec{x})$

$$s^2 = x^\mu x^\nu g_{\mu\nu}$$

Some Notations: $x_\mu = g_{\mu\nu} x^\nu$

Lowering / Raising an index multiplies spatial components by -1

$$g^{\mu}_{\nu} = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{array} \right)$$

Any set of four quantities which transform like the co-ordinates under LT will be called Lorentz vector.

E.g.: $(-E, p_x, p_y, p_z)$ is the energy-momentum vector.
Lorentz Transformations

Lorentz Transform: Set of Linear transformations of (t,x,y,z) which keeps $s^2 = t^2 - x^2 - y^2 - z^2$ invariant.

Linear Transforms: $x'^\mu = \Lambda_\nu^\mu x^\nu$

Lorentz Invariance of $s^2$: $g_{\mu \nu} x'^\mu x'^\nu = g_{\mu \nu} \Lambda_\rho^\mu \Lambda_\sigma^\nu x^\rho x^\sigma = g_{\rho \sigma} x^\rho x^\sigma$

Matrix Form: $s^2 = x^T g x = x' = L x$

$g_{\rho \sigma} = g_{\mu \nu} \Lambda_\rho^\mu \Lambda_\sigma^\nu = \Lambda_{\rho \sigma} \Lambda^\rho_\nu$

Using $g_{\rho \sigma}^\rho = \delta^\rho_\sigma$ $\delta^\rho_\sigma = \Lambda^\rho_\nu \Lambda^\nu_\sigma$ So $(\Lambda^{-1})^\rho_\nu = \Lambda^\nu_\rho$

Taking Det: $\text{Det } g = (\text{Det } L)^2 \text{ Det } g$ $\text{Det } L = \pm 1$

$\text{Det } L = +1$ corresponds to the group $SO(3,1)$. These are called proper Lorentz Transforms, while the transforms with $\text{Det } L = -1$ are the improper Lorentz transforms.
Lorentz Transformations

\( g_{\rho \sigma} = g_{\mu \nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma \)

Considering \( g_{00} \)

\[ 1 = \Lambda_0^\rho g_{\rho \sigma} \Lambda_0^\sigma = (\Lambda_0^0)^2 - (\Lambda_i^i)^2 \]

\[ |\Lambda_0^0| \geq 1 \]

<table>
<thead>
<tr>
<th>Det ( L )</th>
<th>( \Lambda_0^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Proper Orthochronous</strong></td>
<td>+1</td>
</tr>
<tr>
<td><strong>Improper Orthochronous</strong></td>
<td>-1</td>
</tr>
<tr>
<td><strong>Proper Non-Orthochronous</strong></td>
<td>+1</td>
</tr>
<tr>
<td><strong>Improper Non-Orthochronous</strong></td>
<td>-1</td>
</tr>
</tbody>
</table>

Example:

Rotations:

\[ L = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \]

3 \( \times \) 3 Rotation Matrix

Proper Orthochronous

Improper Rotations:

\[ L = \begin{pmatrix} 1 & 0 \\ 0 & RW \end{pmatrix} \]

Improper Orthochronous

\[ W = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]
Lorentz Transformations

Example: Lorentz Boost

\[ L = \begin{pmatrix} \cosh \eta - \sinh \eta \sigma^x & 0 \\ 0 & 1 \end{pmatrix} \quad \text{Proper Orthochronous} \]

While Rotations form a group by themselves, Lorentz Boosts do not form a group by themselves.

Time Reversal

\[ L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{Improper Non-Orthochronous} \]

\[ IO = \text{PO} \times \text{Parity} \quad \text{In 4D} \]

\[ P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

\[ IN = \text{PO} \times \text{Time Reversal} \quad \text{In 4D} \]

\[ T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ PN = \text{PO} \times \text{Time Reversal} \times \text{Parity} \]

Will focus on proper orthochronous Lorentz Transformations from now on

Group \( SO(3,1) \)
Generators of Lorentz Group

Infinitesimal Lorentz Transformations: \[ \Lambda^\mu_\nu = \delta^\mu_\nu + \epsilon^\mu_\nu \]

\[ g_{\rho\sigma} = g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma \]

\[ g_{\rho\sigma} = g_{\mu\nu} (\delta^\mu_\rho + \epsilon^\mu_\rho) (\delta^\nu_\sigma + \epsilon^\nu_\sigma) = g_{\rho\sigma} + g_{\mu\sigma} \epsilon^\mu_\rho + g_{\rho\nu} \epsilon^\nu_\sigma \]

\[ = g_{\rho\sigma} + \epsilon_{\sigma\rho} + \epsilon_{\rho\sigma} \]

\[ \epsilon_{\mu\nu} \text{ is a 4D antisymmetric tensor} \]

6 indep, parameters \[ \rightarrow \] 6 generators of Lorentz Group

Parametrization of Infinitesimal Transforms:

\[ \epsilon = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = a \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \]

\[ U(\Lambda) = 1 - i \epsilon_{\mu\nu} M^{\mu\nu} \]

\[ M^{\mu\nu} = -M^{\nu\mu} \]

\[ +d \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + e \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + f \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \]
Generators of Lorentz Group

By Group Properties

\[ U^{-1}(\Lambda)U(\Lambda')U(\Lambda) = U(\Lambda^{-1}\Lambda') \]

Use

\[ \Lambda'_\mu = \delta^\mu_\nu + \epsilon^\mu_\nu \]

\[ U(\Lambda') = 1 - i\epsilon_{\mu\nu}M^{\mu\nu} \]

LHS

\[ U^{-1}(\Lambda)U(\Lambda')U(\Lambda) = 1 - i\epsilon_{\mu\nu}U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) \]

RHS

\[ (\Lambda^{-1}\Lambda')\rho_\sigma = 1 + (\Lambda^{-1})^\mu_\rho \epsilon_{\mu\nu}\Lambda^\nu_\sigma = 1 + \Lambda^\mu_\rho \epsilon_{\mu\nu}\Lambda^\nu_\sigma \]

\[ U(\Lambda^{-1}\Lambda') = 1 - i\epsilon_{\mu\nu}\Lambda^\mu_\rho \Lambda^\nu_\sigma M^{\rho\sigma} \]

Comparing coeff.

\[ U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) = \Lambda^\mu_\rho \Lambda^\nu_\sigma M^{\rho\sigma} \]

Generators themselves transform as rank 2 Tensors

Transformation Law for vectors

\[ U^{-1}(\Lambda)P^\mu U(\Lambda) = \Lambda^\mu_\nu P^\nu \]
Generators of Lorentz Group

\[ U^{-1}(\Lambda) M^{\mu\nu} U(\Lambda) = \Lambda^\mu_{\rho} \Lambda^\nu_{\sigma} M^{\rho\sigma} \]

Now take \( \Lambda \) to be an infinitesimal transformation.

Expand and compare co-efficients on both sides to get the Lie Brackets

\[ [M^{\mu\nu}, M^{\rho\sigma}] = i [g^{\mu\rho} M^{\nu\sigma} - g^{\nu\rho} M^{\mu\sigma}] - \rho \leftrightarrow \sigma \]

Identify:

- Boost Generators
  \( K_i \equiv M^{i0} \)

- Rotation Generators
  \( J_i \equiv (1/2) \varepsilon_{ijk} M^{jk} \)
Lie Algebra of Lorentz Group

You can explicitly show that \([J_i, J_j] = i\epsilon_{ijk} J_k\)

Easiest way to see this: \[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & d & 0 \\
0 & -d & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
is infinitesimal form of
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos d & \sin d & 0 \\
0 & -\sin d & \cos d & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
which represents rotation about z axis

The transforms corresponding to \(J_i\) is a spatial rotation about axis i. \(J_i\) are the Angular Momentum Operators.

The generators \(K_i\) correspond to infinitesimal Lorentz Boosts with velocity in i direction

Easy to show that
\[
[J_i, K_j] = i\epsilon_{ijk} K_k
\]
\[
[K_i, K_j] = -i\epsilon_{ijk} J_k
\]
SO(3,1) and SU(2) X SU(2)  

Consider the operators

\[
N_i = \frac{1}{2}(J_i + iK_i) \quad N_i^\dagger = \frac{1}{2}(J_i - iK_i)
\]

\[
[N_i, N_j] = \frac{1}{4}[(J_i + iK_i), (J_j + iK_j)] = \frac{1}{4}([J_i, J_j] - [K_i, K_j] + i[K_i, J_j] + i[J_i, K_j])
\]

\[
= \frac{1}{4}(i\epsilon_{ijk}J_k + i\epsilon_{ijk}J_k + \epsilon_{jik}K_k - \epsilon_{ijk}K_k) = \frac{1}{2}i\epsilon_{ijk}(J_k + iK_k)
\]

\[
= i\epsilon_{ijk}N_k
\]

\[
[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger
\]

\[
[N_i, N_j^\dagger] = \frac{1}{4}[(J_i + iK_i), (J_j - iK_j)] = \frac{1}{4}([J_i, J_j] + [K_i, K_j] + i[K_i, J_j] - i[J_i, K_j])
\]

\[
= 0
\]

So, \(N_i\) and \(N_i^\dagger\) independently follow the Lie Algebra of SU(2)
Irreps of Lorentz Group

$N_i$ and $N^*_i$ independently follow the Lie Algebra of SU(2)

E.g. $N_i N_i$ and $N^*_i N^*_i$ are operators with eigenvalue $m(m+1)$ and $n(n+1)$ for half-integer $m$ and $n$

All finite dimensional irreps of the Lorentz Group can be written as $(m/2, n/2)$ where $m/2$ is the $m+1$ dimensional irrep of SU(2)

We will later see a description of Lorentz invariant systems in terms of fields which transform according to different finite dimensional irreps of the Lorentz group

E.g. $(0,0)$ is corresponds to scalars,
   $(1/2,0)$ and $(0,1/2)$ corresponds to (left and right handed) Weyl Spinors
   $(1/2,0) \otimes (0,1/2) = (1/2,1/2)$ gives a spin 1 representation in 4D (vector fields)
Function Space and Lorentz Transformations

Let us study the action of LT on functions of \( t, x, y, z \). i.e. \( \phi (t, x, y, z) \)

i.e. Let the same quantity be described in 2 frames by \( \phi (t, x, y, z) \) and \( \phi' (t', x', y', z') \)

Using \( T[G_a] \phi = \phi' = \phi (T^{-1}[G_a] (x, y, z, t)) \)

\[
\phi' (x') = \phi(\Lambda^{-1} x') \quad \text{Let us consider infinitesimal transformations}
\]

\[
(\Lambda^{-1} x')^\mu = (\Lambda^{-1})^\nu_\mu (x')^\nu = (\delta^\mu_\nu + \epsilon^\mu_\nu (x')^\nu = (x')^\mu + \epsilon^\mu_\nu (x')^\nu = (x')^\mu + g^{\mu \rho} \epsilon_{\rho \nu} (x')^\nu
\]

\[
\phi' (x') = \phi(\Lambda^{-1} x') = \phi(x') + g^{\mu \rho} \epsilon_{\rho \nu} (x')^\nu \partial_\mu \phi(x')
\]

By defn. of generators, \( \phi' (x') = [1 - i\epsilon_{\rho \nu} M^{\rho \nu}] \phi(x) \)

Since \( \epsilon \) is antisymmetric, we should compare the antisymmetric part of its co-eff

\[
M^{\rho \nu} = i[g^{\mu \rho} x^\nu \partial_\mu - g^{\mu \nu} x^\rho \partial_\mu] = -i[x^\rho \partial^\nu - x^\nu \partial^\rho]
\]

This is a infinite dimensional form for the generators, but is it the most general form?