Stability of circular orbits

Sourendu Gupta

TIFR, Mumbai, India

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Angular motion

Since the radial Hamiltonian is $H = p^2/2m + U(r)$, the radial equation of motion is $m\ddot{r} = -dU/dr$. Now, in the notation used earlier, $U(u) = V(u) + u^2$ where $u = \lambda/r$ and $\lambda^2 = |\mathbf{L}|^2/(2m)$. Since $dr = -\lambda du/u^2$, we find

$$\frac{dU}{dr} = -\frac{u^2}{\lambda}\left[\frac{dV}{du} + u\right].$$

The definition of the angular velocity gives

$$\dot{\phi} = \sqrt{\frac{2}{m}}\frac{\lambda}{r^2} = \sqrt{\frac{2}{m}}\frac{u^2}{\lambda}, \quad \text{so} \quad \frac{d}{dt} = \sqrt{\frac{2}{m}}\frac{u^2}{\lambda}\frac{d}{d\phi}.$$

So, denoting derivatives with respect to ϕ by primes, one has

$$m\ddot{r} = m\,\frac{2}{m}\,\frac{u^2}{\lambda}\frac{d}{d\phi}\left\{u^2\frac{d(1/u)}{d\phi}\right\} = -\frac{2u^2}{\lambda}u''.$$

The orbit equations and a special solution

Putting these together, we find the differential equation for the orbit,

$$u'' = -\frac{1}{2}\left[\frac{dV}{du} + u\right].$$

The right hand side is the derivative of the effective potential. We shall abbreviate the right hand side by writing it as $-W(u)$.

A special solution of the orbit equations is obtained if the potential has a minimum. In that case $W(u_0) = 0$ for some zero, and $u'' = 0$ if the separation between the particles is u_0. This is the condition for a circular orbit. Assuming that the potential is attractive and a power law, *i.e.*, $V(u) = -\kappa u^n$, one finds $W(u) = -\kappa n u^{n-1} + u$. The solution for the circular orbit is $u_0^{n-2} = 1/(\kappa n)$.

Problem 37: Circular orbits

Check that for an attractive $1/r$ potential, this condition gives the same result as that obtained earlier.

Small changes in initial conditions

Make a small perturbation of the initial conditions around a circular orbit to $u = u_0 + x$, where x is small. Then $u'' = x''$. The right hand side can be written down as a Taylor expansion of $W(u)$. Then the orbit equation $u'' = -W(u)$ takes the form

$$x'' = -W_1 x - \frac{W_2}{2} x^2 - \frac{W_3}{3!} x^3 + \cdots ,$$

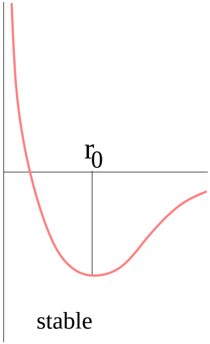
where W_n is the n-th derivative of W with respect to u, evaluated at $u = u_0$.

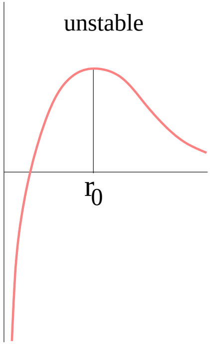
If x is small enough, one might be able to retain only the first term on the right. In this case, the orbit equation reduces to $x'' = -W_1 x$. This has solutions

$$u(\phi) = u_0 + a_1 \mathrm{e}^{i\omega\phi} + b_1 \mathrm{e}^{-i\omega\phi}.$$

As long as $W_1 > 0$ the solution is oscillatory, and the solution oscillates around the initial trajectory.

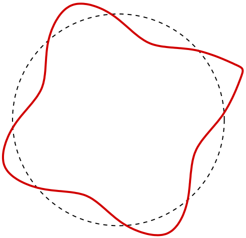
Stability of circular orbits





The stability of circular orbits depends on whether the orbit is at the local minimum or maximum of u. When it is at the minimum, small displacements give rise to simple harmonic oscillations around the stable orbit. This is an extension of the analysis of static stability to a non-inertial frame.

Closed orbits to first order in perturbations



For the attractive power law potentials given earlier, one has

$$W_1 = -\kappa n(n-1)u_0^{n-2} + 1 = -n + 1 + 1 = 2 - n.$$

This is positive for $n < 2$. Note that $\omega = \sqrt{2-n}$, and hence depends only the exponent in the force law.

Clearly, x can increase exponentially for $n > 2$, and hence circular orbits are unstable against perturbations for such potentials.

For $n < 2$, the orbits are closed whenever ω is rational. For the special case of $n = 1$, the orbit oscillates once in the radial direction as it goes once round in the angular direction, *i.e.*, the orbit is elliptical.

Higher order in perturbations

The leading order result can be written as

$$u(\phi) = u_0 \left[1 + \frac{a_1}{u_0} \cos \omega\phi\right],$$

where a_1 is a small quantity of order $\epsilon = \delta u/u_0$. If the orbit is closed then the nonlinear terms generate higher harmonics at higher orders in ϵ. By examining the x^2 term, one sees that a constant and a $\cos 2\omega\phi$ term are generated to order ϵ^2, and from the x^3 term one can see that a third harmonic is generated to order ϵ^3. Beyond the leading order, one may try out a double expansion of the form

$$x = \alpha_0\epsilon^2 + \alpha_1\epsilon \cos \omega\phi + \alpha_2\epsilon^2 \cos 2\omega\phi + \alpha_3\epsilon^3 \cos 3\omega\phi + \cdots$$

Consistency of such a solution can be checked to this order by keeping terms up to x^3, and in each keeping terms up to order ϵ^3 and the third harmonic.

A double expansion

This means we write

$$x^2 = \frac{\alpha_1^2\epsilon^2}{2}[\cos 2\omega\phi+1]+\epsilon^3[2\alpha_0\alpha_1+\alpha_1\alpha_2]\cos\omega\phi+\epsilon^3\alpha_1\alpha_2\cos 3\omega\phi+\cdots,$$

where the neglected terms are either higher powers of ϵ or higher harmonics. Similarly,

$$x^3 = \frac{\epsilon^3\alpha_1^3}{4}[\cos 3\omega\phi + 3\cos\omega\phi] + \cdots$$

From the equation, $x'' = -W_1x - W_2x^2/2 - W_3x^3/3!$, we find to order ϵ, $\omega^2 = W_1$, exactly as before. At order ϵ^2 we find

$$-4\omega^2\alpha_2\cos 2\omega\phi = -W_1\alpha_0 - W_1\alpha_2\cos 2\omega\phi - \frac{W_2}{4}[\cos 2\omega\phi + 1]\alpha_1^2.$$

Equating terms in the same harmonics, we obtain

$$\alpha_0 = -\frac{W_2\alpha_1^2}{4W_1} \qquad \text{and} \qquad \alpha_2 = \frac{W_2\alpha_1^2}{12W_1} = -\frac{\alpha_0}{3}.$$

The cubic term in the perturbation

The cubic terms in the perturbation are interesting. The third harmonic gives a relation between a_3 and a_1, as expected, but the first harmonic gives a consistency relation—

$$0 = -\frac{W_2}{2}[2\alpha_0 + \alpha_2]\alpha_1 \cos\omega\phi - \frac{W_3}{3!}\frac{3\alpha_1^3}{4}\cos\omega\phi.$$

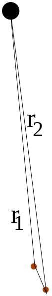
Substituting the values of $\alpha_{0,2}$ into this, one can remove all powers of α_1, and obtain the relation $\boxed{5W_2^2 = 3W_1W_3}$.

With attractive power law potentials, $V(u) = -\kappa u^n$, we have

$$W_1 = \frac{(n-2)}{2}, W_2 = \frac{(n-1)(n-2)}{2u_0}, W_3 = \frac{(n-1)(n-2)(n-3)}{2u_0^2}.$$

For $n = 1$ $W_2 = W_3 = 0$, so the relation is satisfied. Otherwise, $5(n-1) = 3(n-3)$, *i.e.*, $n = -2$. This is Bertrand's theorem: generic orbits are not closed when $n \neq 1$ or -2. Explicit solutions show orbits are closed for these two cases.

The origin of tidal forces



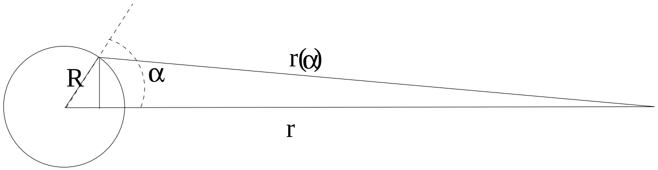
Consider two bodies each moving under the action of the force of a third body. The positions $\mathbf{x}_1$ and $\mathbf{x}_2$ then change only due to the interaction $V(x_1)$ and $V(x_2)$. If these two bodies exert no forces on each other, their relative position $\mathbf{x}_{12} = \mathbf{x}_1 - \mathbf{x}_2$ is nevertheless subject to a tidal force

$$\ddot{\mathbf{x}_{12}} = -\nabla_1 V(x_1) + \nabla_2 v(x_2).$$

Tidal forces can arise from any law of force.
Assume that $V(r) = -\kappa r^n$, so that the force is $\mathbf{F}(r) = -\kappa n r^{n-1}\hat{\mathbf{r}}$. If the position vector $\mathbf{x}_2$ is assumed to depend on a parameter α, then one can write the tidal forces as

$$F_t(r) = -\kappa n \left[(n-1) r^{n-2} \hat{\mathbf{r}} \frac{dr}{d\alpha} + r^{n-1} \hat{\phi} \frac{d\phi}{d\alpha} \right].$$

Tidal geometry of a sphere



The typical geometry of tides is as above, with $\epsilon = R/r \ll 1$. We find that

$$r^2(\alpha) = r^2\left[1 - 2\epsilon\cos\alpha + \epsilon^2\right], \qquad \tan\phi = \frac{\epsilon\sin\alpha}{1-\epsilon\cos\alpha}.$$

To leading order in ϵ one finds

$$\frac{dr}{d\alpha} = -2R\sin\alpha, \qquad \frac{d\phi}{d\alpha} = \epsilon\cos\alpha.$$

As a result, one finds the tidal force

$$\mathbf{F}_t = -\kappa R n r^{n-2}\left[-2\sin\alpha\hat{\mathbf{r}} + \cos\alpha\hat{\phi}\right].$$

Some problems

Problem 38: Tide shapes

Transform the expression for tidal forces to a system of spherical coordinates fixed to the center of the sphere on which the tides are being studied. Does this explain why ocean tides have periodicity of about 12 hours?

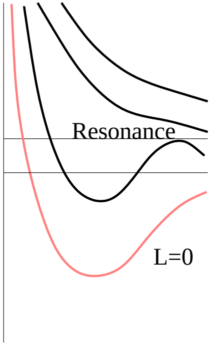
Problem 39: Three body tides

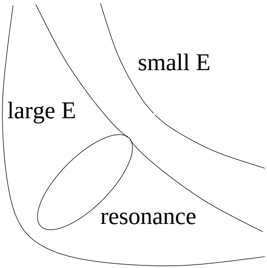
Consider the tides on a sphere due to two external fixed centers of force. Assume that the angle between the lines joining the center of the sphere to the two external centers of force make an angle of ψ. How does the shape of the tides depend on ψ?

Problem 40: Roche limit

Read about Roche's limit and work out the physics.

Ubiquitious scattering problems





Problem 40: Dynamics near a resonance

Use perturbation theory to work out the dynamics near a resonance.

Keywords and References

Keywords

equation for the orbit, circular orbit, small perturbation, unstable against perturbations, Bertrand's theorem, tidal force, perturbation theory, resonance

References

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