Central Forces

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Classical Mechanics 2011
September 5, 2011

The center of mass

The center of mass (CM) of N particles, the i-th of which has mass m_i and position $\mathbf{x}_i$ is defined by the relation

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{x}_i \qquad \text{where} \qquad M = \sum_{i=1}^{N} m_i .$$

The velocity of the CM is therefore

$$\mathbf{V} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{v}_i = \frac{\mathbf{P}}{M},$$

where $\mathbf{P}$ is the net momentum of the system.

If the particles move only under their mutual interactions, then the dynamics of the system must be invariant under translations of the CM. Since the Lagrangian does not depend on $\mathbf{R}$, $\mathbf{P}$ is conserved, *i.e.*, $\mathbf{V}$ is constant. It is possible to choose an inertial frame (the CM frame) in which $\mathbf{R} = 0$ and $\mathbf{V} = 0$.

Newton's third law of motion

Newton's third law of motion is a consequence of translation invariance, if one assumes the equations of motion. Assume that the force exerted on particle i by particle j is $\mathbf{f}_{ij}$, then

$$\dot{\mathbf{V}} = \frac{1}{M}\sum_{i=1}^{N} \dot{\mathbf{p}}_i = \frac{1}{M}\sum_{ij=1}^{N} \mathbf{f}_{ij}.$$

Since $\dot{\mathbf{V}}$ vanishes no matter what the forces are, the only general solution is that $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$.

Translation invariance is a more primitive concept. When the forces between two bodies are not equal and opposite to each other, one would first assume that the equations of motion be modified rather than give up translation invariance. Only if this does not explain the observed phenomena would one tamper with the symmetries of space-time.

Conservation of angular momentum

Invariance under rotations implies that the dynamics of a system of N particles moving only under their mutual interactions cannot change if the system is rotated as a whole. For a rotation around the axis $\hat{\mathbf{n}}$ by an angle $\delta\psi$, position vectors $\mathbf{x}_i$ are changed by $\delta\mathbf{x}_i = \delta\psi\hat{\mathbf{n}} \times \mathbf{x}_i$, and similar exressions hold for other vectors. Since the Lagrangian is invariant,

$$0 = \sum_{i=1}^{N} \left[\frac{\partial L}{\partial \mathbf{x}_i} \cdot \delta r_i + \frac{\partial L}{\partial \mathbf{p}_i} \cdot \delta p_i \right] = \delta\psi \sum_{i=1}^{N} [\dot{\mathbf{p}}_i \cdot \hat{\mathbf{n}} \times \mathbf{x}_i + \dot{\mathbf{x}}_i \cdot \hat{\mathbf{n}} \times \mathbf{p}_i] \,.$$

Since $\delta\psi$ is totally arbitrary, one can write

$$0 = \hat{\mathbf{n}} \cdot \sum_{i=1}^{N} [\mathbf{x}_i \times \dot{\mathbf{p}}_i + \dot{\mathbf{x}}_i \times \mathbf{p}_i] = \hat{\mathbf{n}} \cdot \frac{d}{dt} \sum_{i=1}^{N} \mathbf{x}_i \times \mathbf{p}_i = \hat{\mathbf{n}} \cdot \frac{d}{dt}\mathbf{L}.$$

Since $\hat{\mathbf{n}}$ is completely arbitrary, this shows that the angular momentum $\mathbf{L}$ is conserved.

Two particle systems

The dynamics of a two particle system requires the specification of two position vectors and two momenta at all times. However, if they move only under their mutual interactions, then the CM $\mathbf{R}$ is a cyclic coordinate and completely fixed by initial conditions. The solution of the system then requires the knowledge only of the remaining two vectors, *i.e.*, of 6 coordinates in phase space. Define $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$. Using this along with $\mathbf{R} = (m_1\mathbf{x}_1 + m_2\mathbf{x}_2)/M$, we can write

$$\mathbf{x}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{x}, \qquad \mathbf{x}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{x}.$$

Then, for the kinetic energy, we find

$$2T = m_1\dot{\mathbf{x}}_1^2 + m_2\dot{\mathbf{x}}_2^2 = M\dot{\mathbf{R}}^2 + \frac{m_1 m_2}{M}\dot{\mathbf{x}}^2 = m\dot{\mathbf{x}}^2 = \frac{1}{m}\mathbf{p}^2.$$

where the last equalities hold in the CM frame with reduced mass $m = m_1 m_2/M$, and conjugate momenta $\mathbf{p}$.

Central forces

The Lagrangian for two particles moving under their mutual interactions is

$$L = \frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x}),$$

where $V(\mathbf{x})$ is the potential between them. Rotation invariance requires $V(\mathbf{x}) = V(|\mathbf{x}|)$. This is a central force problem. Then $\mathbf{L}$ is conserved, where

$$\mathbf{L} = \mathbf{x}_1\times\mathbf{p}_1+\mathbf{x}_2\times\mathbf{p}_2 = \mathbf{R}\times\mathbf{P}+\mathbf{x}\times\left(\frac{m_2}{M}\mathbf{p}_1 - \frac{m_1}{M}\mathbf{p}_2\right) = \mathbf{x}\times(m\dot{\mathbf{x}}) = \mathbf{x}\times\mathbf{p}.$$

Problem 31: Velocity dependent forces

For velocity dependent forces, rotational invariance allows $V(|\mathbf{x}|, |\mathbf{v}|, \mathbf{x}\cdot\mathbf{v})$. In this general case check whether $\mathbf{L}$ is conserved. What does this tell you about the motion of a charge in a magnetic monopole field?

The degenerate case $\mathbf{L} = 0$

When $\mathbf{L} = 0$, $\mathbf{x}$ and $\mathbf{p}$ are collinear, so the motion is 1-dimensional. Choose this to be along the x direction. The Hamiltonian, $H = p^2/(2m) + V(x)$ is time independent; so the energy $E = H$ is conserved. We can solve this by quadrature

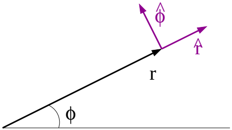
$$t = \int_{x_0}^{x} \frac{dx}{\sqrt{2[E - V(x)]/m}}, \qquad \text{where} \qquad x(t = 0) = x_0.$$

Problem 32: Power laws

Assume that the potential $V(x) = x^\alpha$. What are the conditions that trajectories are (a) open and (b) closed? What are the conditions that all trajectories are closed? Solve the problem of two particles moving in such force fields with $\mathbf{L} = 0$.

Polar coordinates

In the general case of motion in central forces, the conservation of $\mathbf{L}$ forces $\mathbf{x}$ and $\mathbf{p}$ to be in the plane perpendicular to $\mathbf{L}$. Hence, 2 of the 6 coordinates of phase space are forced to be zero. The motion is restricted to the remaining 4-dimensional phase space. Even so, one of the coordinates is cyclic, so the motion will eventually turn out to be effectively one-dimensional.



$$\dot{\hat{\mathbf{r}}} = \dot\phi\hat\phi$$
$$\dot{\hat\phi} = -\dot\phi\hat{\mathbf{r}}$$

It is common to turn to polar coordinates next. Define unit vectors in the radial direction, $\hat{\mathbf{r}}$, and the orthogonal direction, $\hat\phi$. Now any vector $\mathbf{x} = r\hat{\mathbf{r}}$. As a result, $\dot{\mathbf{x}} = \dot r\hat{\mathbf{r}} + r\dot\phi\hat\phi$, and

$$\begin{aligned} \mathbf{L} &= mr^2\dot\phi\,\hat{\mathbf{r}}\times\hat\phi, \\ T &= \frac{m}{2}\dot{\mathbf{x}}^2 = \frac{m}{2}\left(\dot r^2 + r^2\dot\phi^2\right). \end{aligned}$$

Reducing the problem to one dimension

Since the coordinate ϕ is cyclic, the Hamiltonian becomes effectively one-dimensional—

$$H = \frac{m}{2}\dot{r}^2 + \frac{|\mathbf{L}|^2}{2mr^2} + V(r) = \frac{m}{2}\dot{r}^2 + U(r), \quad \text{where} \quad U(r) = V(r) + \frac{|\mathbf{L}|^2}{2mr^2},$$

where $U(r)$ is called the effective potential. Further, since the Hamiltonian is time-independent, one can solve this completely by quadrature

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{2[E - U(r)]/m}}.$$

Problem 33: Power laws

Assume that the potential $V(x) = x^\alpha$. What are the conditions that trajectories are (a) open and (b) closed? What are the conditions that all trajectories are closed? Solve the problem of two particles moving in such force fields with $\mathbf{L} = 0$.

Keywords and References

Keywords

center of mass, invariant under translations, the CM frame, Newton's third law of motion, primitive concept, invariance under rotations, reduced mass, central force problem, polar coordinates, effective potential

References

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