Spinor Representation of Lorentz Group

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It is interesting to see how the concept of spinors can be introduced by simply studying the homomorphism between the proper orthochronous Lorentz group and the group of $SL(2, C)$ matrices (or, equivalently , between proper rotation group $O^+(3)$ and $SU(2, C)$ group).

We begin by showing the well known homomorphism between $O^+(3)$ and the $SU(2, C)$ group which gives a relation

$$R_{ij} = \frac{1}{2} Tr(\sigma_i \; Q \; \sigma_j Q^\dagger) \tag{1}$$

where R_{ij} is a typical element of the rotation matrix, σ_is are Pauli matrices and Q is a unimodular, unitary 2×2 complex matrix.

The relation clearly reveals the two-to one homomorphism between the two groups. It is easy to verify that this relation is only obeyed by proper rotation matrices.

Analogously we can show a similar two-to one homomorphism between the Lorentz group and the $SL(2, C)$ matrices which lead to the equation

$$L^\mu_\nu = \frac{1}{2} Tr(\sigma^\mu \; A \; \sigma_\nu \; A^\dagger) \tag{2}$$

where L^μ_ν is an element of the Lorentz matrix, σ_μs are the pauli matrices, A is unimodular 2×2 complex matrix.

The unimodular complex matrices form a two dimensional representation of the Lorentz group and we define spinors as vectors in this two dimensional complex representation space, obeying the following transformation equation

$$\xi^{(1)} = A\xi, det \; A = +1 \tag{3}$$

A two dimensional scalar in the above space is proposed to be of the form $\xi^\alpha C_{\alpha\beta} \eta^\beta$ where C is the metric of this space. Hence we can derive the following equation for C

$$A^T \; C \; A = C, \tag{4}$$

which is satisfied by both symmetric and antisymmetric matrices. However, we can easily show that the symmetric solution for C is the null 2×2 matrix, so that we choose a convenient antisymmetric form that is the 2×2

antisymmetric matrix with 1 and -1 as the offdiagonal terms. This is called the metric spinor.

From this point it is trivial to construct higher order spinors which transform in a way exactly analogous to Lorentz tensor

$$\xi^{(1)\alpha\beta} = A^{\alpha}_{\gamma} \times A^{\beta}_{\delta} \times \xi^{\gamma\delta} \tag{5}$$

Though not a part of the talk, we observe that proceeding in the same way we can work out a general connection between spinors and Lorentz tensors.

References

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