SPINOR REPRESENTATIONS OF LORENTZ TRANSFORMATIONS

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My aim here is at introducing the concept of *Spinors* to beginners; who have worked with vectors (not even tensors) and have an idea that matrices are linear transformations. *Spinors* are analogous to vectors and tensors but with different transformations properties.

We first understand the correspondence between the 3 dimensional rotation group $O(3)$ and the $SU(2,C)$ group. A rotation takes a spatial point to another without changing the length (norm invariance). A rotation matrix has 3 parameters (3 Euler angles) to be determined. It can be seen that a 2×2 complex matrix with the conditions that it is unitary and unimodular ($i.e. SU(2,C)$) also has 3 paremeters to be determined. Consider a Hermitian matrix P. Expressing P as linear combination of the basic Pauli matrices σ_i's i.e $P = x_i\sigma_i$. The coefficients x_i's are the spatial coordinates. Taking a similarity transformation $P' = QPQ^\dagger$ and taking determinant on both sides, we see that the norm is preserved. Thus, this similarity transformation on P has actually behaved like a rotation. So, we say that Q is a $SU(2,C)$ representation of the corresponding $O(3)$ matrix. Worthnoting is that taking $(-Q)$ instead of Q corresponds to the same transformation.

We now proceed as above and generalise it in case of $SL(2,C)$ and $L_+^\uparrow$ - proper orthochronous Lorentz group. A Lorentz transformation has 6 parameters to be determined and a $SL(2,C)$ matrix (with the property that Det=1) too has 6 parameters that characterize it. Consider a Hermitian matrix $X = x^\mu\sigma_\mu$ where x^μ's are components of 4 vector. Take the transformation $X\prime = AXA^\dagger$ where $A \in SL(2,C)$. Here too,taking determinant on both sides we see that the norm is invariant just as in a Lorentz transformation. Further this corresponds to the proper orthochronous transformation and both $A\&(-A)$ correspond to same transformation. Thus A is a $SL(2,C)$ represenation of the $L_+^\uparrow$ transformation.

These $SL(2,C)$ representation matrices act on elements of 2 dimensional complex vector space known as *Spinors* analogous to the 4-vectors which are acted on by Lorentz matrices. Similar to G - the metric tensor for the Lorentz transformations, we have a metric spinor C which satisfies $A^tCA = C$. Interesting to note is that C can be any antisymmetric matrix.

References

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