Effective Theories are Dimensional Analysis

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Effective Field Theories
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Outline

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Units and dimensions

For relativistic field theories we will use natural units $\hbar = 1$ and $c = 1$, and choose to use dimensions of mass for everything. Then every quantity has dimensions which are some power of mass: $[m] = 1$, $[x] = -1$, $[t] = -1$.

Since we need to use $\exp[-iS]$ to define the path integral, the action S must be dimensionless, *i.e.*, $[S] = 0$. Since

$$S = \int d^4x \mathcal{L},$$

we have $[\mathcal{L}] = 4$ in a relativistic theory. For a scalar field $\mathcal{L}_{\rm kin} = \partial_\mu\phi\partial^\mu\phi/2$, so $[\phi] = 1$. For a fermion field, the mass term in the Lagrangian density is $m\overline{\psi}\psi$, so $[\psi] = 3/2$. Finally, in a gauge theory $\mathcal{L}_{\rm kin} = F^{\mu\nu}F_{\mu\nu}/4$, so $[F] = 2$. Since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \cdots$, we must have $[A] = 1$.

Old view: need to banish infinities

Popular view: quantum field theory has infinities such as the self-energy of the electron. Caused by the fact that the field of the electron acting on itself has divergences. These must be removed.

Actually: this has nothing to do with quantum field theory. It already exists in classical models of this kind, such as Lorentz's theory of the electron.

Traditional view: the "bare mass" of the electron is infinite, and it is cancelled by the divergence caused by the self-energy when the interaction is switched on.

Modern view: there is no "bare mass", since the electron is charged. Its charge and mass may depend on the length scale at which we probe it. Classical theory is simply wrong.

Modern view: use only what is known

There is a maximum energy, Λ, available for experiments at any time. The mass scales probed using this are $m \leq \Lambda$, length scales are $\ell \geq 1/\Lambda$. Physics at shorter length scales or larger energy scales is unknown.

Non-relativistic particle of momentum p, mass m, scatters off a short-range potential. Details of potential at energy scales larger than $p^2/(2m)$ or length scales smaller than $1/p$ are unknown. Cannot distinguish:

Problem 1.1

Compute the S-matrix for these three cases in the limit $p \to 0$ and check that it is universal, *i.e.*, independent of the potential. What if the well changed to a barrier? What is the essential physics?

The modern view of QED

For a given Λ, we can write down all the terms we need to describe electrons and photons at smaller energy. For example—

$$\mathcal{L} = c_3\Lambda\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}(\partial\!\!\!/ - c_4 A\!\!\!/)\psi + \frac{c_5}{\Lambda}\overline{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu} + \frac{c_6}{\Lambda^2}(\overline{\psi}\psi)^2 + \frac{c_8}{\Lambda^4}(F_{\mu\nu}F^{\mu\nu}) + \cdots$$

The powers of Λ are chosen to keep c_i dimensionless. As $\Lambda \to \infty$, the effect of the terms with it in the denominator will become smaller; such terms are called irrelevant. Terms with positive powers of Λ will become more important; these are called relevant. Terms involving c_4 are called marginal.

In the limit $\Lambda \to \infty$, the irrelevant terms drop away and only the marginal and relevant terms remain. However, keeping the low energy theory fixed involves fine tuning the irrelevant couplings. This may be construed as a problem.

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Rayleigh scattering of light

Rayleigh scattering of light on atoms deals with frequencies, $\omega \ll \Lambda$, where $\Lambda \simeq 10$ eV is the energy required for electronic transitions ($\omega \simeq \Lambda$ for mid-UV light). So, we can try to describe this scattering process in an effective field theory.

Atoms are non relativistic and uncharged, so for the atoms (mass $M \gg \omega$) we write the usual non-relativistic action

$$\mathcal{L}_{\rm kin} = \phi^* \left(i\frac{\partial}{\partial t} - \frac{p^2}{2M} \right) \phi,$$

where p is the momentum of the atom. Since the operator within brackets has dimensions of energy, we have $[\phi] = 3/2$.
Since atoms are not created or destroyed, the interaction terms must contain the product $\phi^*\phi$. This has dimension 3. Since scattering takes place from neutral atoms, we cannot use the prescription $p \to p - eA$. So the coupling can only involve powers of $F_{\mu\nu}$.

Organizing terms by dimension

Lorentz invariance then forces us to contract indices among the factors of the EM field tensor, or with the 4-momentum of the atom, p_μ. The lowest dimensional terms have mass dimension 7:

$$\mathcal{L}_{\rm int}{}^7 = c_7^1 \phi^* \phi F_{\mu\nu} F^{\mu\nu} + c_7^2 p_\mu \phi^* p_\nu \phi F^{\mu\nu}$$

so that $[c_7^i] = -3$. Since the only long distance scale is the size of the atom, a, it is natural to write $c_7^1 \simeq a^3$. The second term vanishes in the limit when $v = p/M \ll 1$.

The Born scattering cross section must be proportional to a^6. As a result, dimensional analysis gives

$$\sigma \propto a^6 \omega^4,$$

implying that blue light scatters more than red.

Problem 1.2

Compute the Born cross section within this effective theory.

Corrections to the cross section

One can try to improve the accuracy of the predictions by including higher dimensional terms of the kind

$$\mathcal{L}_{\rm int}{}^9 = c_9^1 \phi^* \phi F_{\mu\nu} F^\nu_\lambda F^{\lambda\mu} + \cdots$$

Now $[c_9^1] = -5$, and the effect of this term has to be smaller by powers of Λ if the dimension 7 terms are to remain the most important term.

We achieve this by writing $c_9^1 \simeq a^3/\Lambda^2$. Higher order terms will have to be suppressed by higher powers of Λ. If we started from a more fundamental theory, then we would be able to see these powers of Λ appearing.

The correction to the cross section due to this term is

$$\mathcal{O}\left(\frac{\omega}{\Lambda}\right)^2,$$

and can be neglected as long as $\omega \ll \Lambda$.

W, Z and their couplings to fermions

For the weak interactions we can collect fermions into doublets

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} \qquad \begin{pmatrix} e \\ \nu_e \end{pmatrix} \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix} \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}$$

The interaction between W and fermions is given by

$$\mathcal{L}_{\rm int} = J^\mu W_\mu, \quad \text{with} \quad J^\mu = -\frac{ig}{\sqrt{2}} V_{ij} \overline{U}_i \gamma^\mu \Pi_L D_j,$$

where V is the Cabibbo-Kobayashi-Maskawa mixing matrix, Π_L is the projection on to left-handed helicities of Dirac fermions, g is the weak coupling, U_i is the field for the up-type fermion in generation i and D_j for the down type fermion in generation j. Since W is a gauge boson with its usual kinetic term, $[g] = 0$.

Decay amplitude

A typical weak decay amplitude is given in the Feynman gauge by

$$A = \left(\frac{ig}{\sqrt{2}}\right)^2 J^\mu J^\nu \left(\frac{-ig_{\mu\nu}}{p^2 - M_W^2}\right),$$

where p is the momentum transfer between the fermion legs, and each of the two fermion currents can be either hadronic or leptonic.

When $p \ll M_W$, the propagator can be Taylor expanded to give

$$\frac{1}{p^2 - M_W^2} = -\frac{1}{M_W^2} - \frac{p^2}{M_W^4} + \cdots$$

To leading order, the amplitude can be captured into

$$\mathcal{L}_{\rm int} = -2\sqrt{2} G_F J_\mu J^\mu, \quad \text{where} \quad 2\sqrt{2} G_F = \frac{g^2}{2M_W^2}.$$

Since $[J] = 3$, we find $[G_F] = -2$. G_F is given by matching the amplitude between the effective theory and SM at Born level.

Using the Fermi theory

This effective Fermi theory of β-decay, is treated in elementary textbooks, so detailed computations will not be done. Recall that the muon lifetime is given by

$$\tau_\mu^{-1} \propto G_F^2 m_\mu^5,$$

where G_F^2 comes from the square of the amplitude and the remainder comes from the 3-body decay phase space, since m_μ is the only scale in the decay. This formula can be used for τ, charm and bottom, but not for the top, since $m_t > M_W$.

Note A: The computation is clearly not improved by going beyond tree level in the effective theory, since that will not reproduce loops in the SM. Loop corrections in effective theories will be taken up later.

Note B: Low-energy experiments are sufficient to discover the V$-$A structure of the current (*i.e.*, the factor of Π_L).

Higher dimensional terms

The Fermi theory was first obtained as a brilliant piece of phenomenology. It remained incredibly successful until experimentally available energies came close to the cutoff $\Lambda \simeq M_W$.

We saw that the Fermi theory could be obtained by expanding a Born amplitude of the SM in powers of p/M_W. Clearly the low-energy effective theory can be corrected by including these terms systematically. Replacing such a non-local term by a sum of many local terms is called the operator product expansion.

Each momentum will be obtained by a derivative, so we will have dimension 8 terms of the kind,

$$\mathcal{L}_{\rm int}{}^{8} = c_8 p_\mu J_\nu p^\mu J^\nu,$$

where $[c_8] = -4$. Tree-level matching of amplitudes then shows that the extra powers of mass will arise as $1/M_W^2$.

The renormalizable standard model

As usual, the kinetic and mass terms in the standard model are of mass dimension 4. The W and Z couplings are of the form described already, and hence of dimension 4. The Yukawa terms are also of dimension 4. The exceptions are the Higgs mass term (dimension 2) and the fermion mass terms (dimension 3).

This is forced by the power counting arguments which we have already seen:

1. Terms in which the dimensions of the product of field operators is less than D are super renormalizable. These couplings are relevant.
2. When the operator dimension is D the term is renormalizable, the coupling is marginal.
3. When the operator dimension is greater than D the term is non-renormalizable, and the coupling is irrelevant.

Every relevant term is problematic

The standard model was developed as an unique renormalizable theory which explained all the data available in the 1970s. Today it is tested up to energy scales of about 1 TeV and found to work. However, we can examine it as an effective theory in order to identify our inadequate understanding of it.

No principle prevents us from adding the trivial super-renormalizable operator 1 to any theory. The dimension 4 coefficient is $c\Lambda^4$, where c is a dimensionless number. Since the SM works for $\Lambda \simeq 1$ TeV, this term, which is the cosmological constant, should be around a TeV^4. However, it is known to be around a meV^4.

This can only be done by fine-tuning $c = 10^{-15}$. This is just so; we know of no theoretical mechanism to achieve this.

The strong CP problem: a marginal term

A possible dimension 4 operator involves the gluon field strength—

$$\mathcal{L}_{\text{int}}{}^{4} = \theta \epsilon^{\mu\nu\lambda\rho} G_{\lambda\rho} G_{\mu\nu} = \theta \widetilde{G}^{\mu\nu} G_{\mu\nu}.$$

This term violates CP, and can be detected experimentally.

Electric dipole moments (EDM) are not invariant under the transformation P. Operator expectation values in charge neutral state are invariant under C. So, if a neutral particle has EDM, then it is not invariant under CP.

The neutron's EDM is measured to be zero within experimental precision (less than 10^{-26} e-cm), implying that θ is zero. Since there is no symmetry reason for this in the SM, it implies a fine-tuning of this coupling. Should we be happy with fine-tuning or look for reasons?

Naturalness and the Higgs mass

The Higgs mass term is a dimension 2 operator $m^2 H^\dagger H$. Since this is relevant, our power counting tells us that the scaling of the mass is $m^2 = c\Lambda^2$. Now, the natural scale of the cutoff is the scale at which new physics kicks in. So, given that the Higgs mass is found to be $\mathcal{O}(100\mathrm{GeV})$, one expects that $\Lambda \simeq 1$ TeV, since the natural scale of dimensionless numbers is of the order of 1. So it is natural to expect that there is physics beyond the SM at this scale.

However, if the new physics arises at much higher scale (say $\Lambda \simeq 10^{15}$ GeV) then one could accommodate it by tuning $c \simeq 0$. This is exactly like the strong CP problem, and would give rise to another fine-tuning problem in the SM.

Problem 1.3

Compute the neutron EDM in terms of θ and the Higgs mass in terms of c and $\Lambda \simeq 10^{15}$ GeV. Which is more fine-tuned, θ or c?

Fermions masses are protected

Fermion mass terms are of the form $m\overline{\psi}\psi$, which has operator dimension 3. As a result, $m = c\Lambda$. Since some of the masses are around MeV and Λ could be potentially as large at 10^{15} GeV, this could give us another fine-tuning problem.

Chiral symmetry prevents this. If we ask for the action to be invariant under the transformation $\psi \to \exp[iz\gamma_5]\psi$ then the operator product $\overline{\psi}\psi$ is not invariant under this transformation, and we must set $c = 0$ in order to preserve this symmetry.

Fermion masses can arise through the Yukawa coupling between the Higgs field and fermions, $\mathcal{L}_{\rm int} = Y\overline{\psi}\psi H$, once H gets a vacuum expectation value (vev). Since this term is renormalizable, it does not require fine-tuning.

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The incredible usefulness of dimensional arguments

Undergrad uses of dimensional arguments are in analysis, mainly in checking formulæ.

A deeper use is for quick understanding of physical phenomena. This rests on the assumption that all scales which arise from the same physical cause are roughly similar. In atomic physics the Rydberg sets a scale for electronic transitions ($R \simeq 10$ eV), and all energy scales are similar to it.

As a result, dimensional arguments become a tool of discovery. When there is a mismatch of scales, there is new physics at work. In molecular spectra one finds scales of meV, due to new physics— vibrational states. At intermediate energy scales one finds new physics— Rayleigh scattering. This use of dimensional analysis also underlies the modern understanding of renormalization.

Is fine-tuning a problem?

From this point of view, fine-tuning is a problem. In the 1970s 't Hooft discovered the disparities of scale in particle physics. For 40 years physicists assumed that this would lead to new physics at the 1 TeV scale. The LHC has not yet shown any new physics. This could become a potential problem in our understanding of field theories.

We could live with it. In atomic physics there is a fine tuning problem. The natural energy scale should be the reduced mass of the electron, which is $m \simeq 1$ MeV. So there is fine-tuned scale $R/m \simeq 10^{-5}$, which we call α^2.

Our understanding of this fine tuning is resolved in the SM where we understand that m is explained through Yukawa couplings. Even so, the hierarchy of mass scale between t and e is achieved through a fine-tuning of dimensionless couplings, which we accept.

The anthropic principle and the landscape

The anthropic solution: the universe is as it is because there are physicists!

Physicists do not create the universe with their models. The laws of physics have to be such that they account for physicists: the universe must be old enough, complex enough, *etc.*.

Landscape of the multiverse: there are uncounted string vacua; universes corresponding to each of these can accomodate all possible physics of the Higgs. The existence of physicists selects the vacuum which accomodates them.

Objections? Counting [Strassler, 2014]: since technicolor, SUSY, small Yukawa, *etc.*, are all possible as solutions of the hierarchy problem, they must exist in the landscape. Begs the question of why physicists arise only in the non-generic string vacua with SM particle content and light Higgs.

Beyond landscape and philosophy

Convert to a physics question: are there many different mechanisms for fine-tuning? Two of many in nuclear physics:

1. Natural cross section for low-energy scattering of nucleons is $1/m_\pi^2 \simeq 20$ mb. Measured value: 40 mb for np, but 300 mb for pp. Lattice computations at generic m_π do not yield large pp cross sections.
2. Carbon is produced in supernovæ, through the process $3^4\mathrm{He} \to {}^{12}\mathrm{C}$. Insufficient ${}^{12}C$ unless there is a resonant state $2^4\mathrm{He} \to {}^8\mathrm{Be}$. Predicted before observations. This famous Hoyle coincidence, was the origin of the anthropic principle.

Only 3 free parameters in QCD: light quark masses. Can these accommodate all nuclear fine-tuning? There are many examples of molecular fine-tuning. What about these? Studying low-energy effective theories could help us to understand whether there are chains of coincidences, or whether each fine-tuning is a separate puzzle.

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Keywords and References

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Natural units; universality; irrelevant, relevant and marginal couplings; fine-tuning problem; Fermi theory of β-decay; low-energy effective theory; operator product expansion; super-renormalizable, renormalizable and non-renormalizable theories; naturalness; cosmological constant; strong CP problem; Higgs mass; chiral symmetry; landscape; Hoyle coincidence; anthropic principle.

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