Test 1

Statistical Mechanics, 2002.
Date: 29th Apr, 2002
Time: 120 minutes.

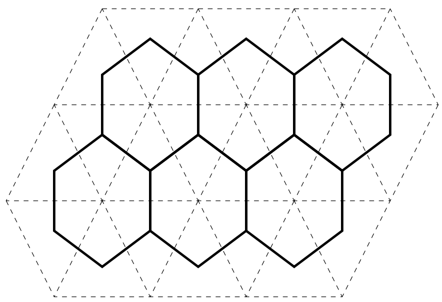
(The relative weight of each question is given in brackets. Number the pages of your answer sheets. Write your name on the first page. Any book or note can be consulted, however, exchange of books, notes or other material is not allowed during the exam.)

1. At the Curie point of metallic iron the spin-spin correlation function falls off as a power of the distance r, *i.e.*, as $1/r^p$. What is the value of p? (5)

   In terms of the critical exponent η, *we have* $p = d - 2 + \eta$. *Now Josephson's law gives* $\nu = (2-\alpha)/d$. *Then Fisher's law gives* $\eta = 2 - \gamma/\nu = 2 - d\gamma/(2-\alpha)$. *Finally* $p = d[1 - \gamma/(2-\alpha)]$. *Since* $d = 3$, $\alpha = 0.12$ *and* $\gamma = 1.33$ *(see Table 1 of Lecture 1)*, $p = 0.88$.

   Marking scheme: *3 points for getting* η, *2 more for getting* p.

2. Sketch the lattice dual to the 2-dimensional triangular lattice. (5)



   Marking scheme: *All or nothing.*

3. A series for the magnetic susceptibility of some material is

$$\chi(z) = 1 + \frac{7}{4}x + \frac{45}{32}x^2 + \frac{161}{128}x^3 + \frac{2387}{2048}x^4 + \frac{9009}{8192}x^5 + \mathcal{O}(x^6)$$

List the orders of all the Padé approximants which you can construct for the logarithmic derivative of this series. (There is no need to actually find the Padé approximants.) (5)

The logarithmic derivative can be computed to order $\mathcal{O}(x^5)$. Then, we can determine all (n, d) Padé approximants with $n + d \leq 4$, i.e., $(0,0)$, $(1,0)$, $(0,1)$, $(2,0)$, $(1,1)$, $(0,2)$, $(3,0)$, $(2,1)$, $(1,2)$, $(0,3)$, $(4,0)$, $(3,1)$, $(2,2)$, $(1,3)$, $(0,4)$.

Marking scheme: *All or nothing.*

4. An ideal gas undergoes Bose condensation. What is the fraction of energy carried by the condensate? (5)

Zero, since the energy of the zero mode vanishes.

Marking scheme: *All or nothing.*

5. Consider an ideal massless, relativistic, Fermi gas at temperature T and fugacity z in a volume V. Expand the expectation value of the energy in powers of the fugacity and write the first two terms in the expansion. (Assume that each level has degeneracy g.) (10)

The energy of the ideal Fermi gas is

$$\langle E \rangle = gV \int_0^\infty \frac{d^3pE}{z^{-1}\exp(E/T) + 1} = 4\pi gVT^4 \int_0^\infty \frac{dx x^3}{z^{-1}\mathrm{e}^x + 1}$$

where $x = E/T$. Writing this in terms of Fermi integrals, we have

$$\frac{\langle E \rangle}{4!\pi gVT^4} = F_4(z) = z - \frac{z^2}{16} + \cdots$$

where the expansion is given in Problem set 5.2.2.a.

Marking scheme: *2 points for writing the first form of the integral, 2 more for integrating over angles, 4 more for converting it to dimensionless form, 2 more for the correctly writing out the expansion.*

6. A Markov chain is to be set up to simulate the probability distribution on the state space $\{-1, 0, 1\}$ with all moments of the distribution vanishing. Write down the matrix of Markov transition probabilities with the lowest thermalisation time? (Use the convention that columns add up to unity.) (10)

The probability distribution is $p(-1) = p(1) = 0$ and $p(0) = 1$. Using the convention above, the required transition matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

No matter what the state of the system at the initial time, in the very next time step the state becomes 0. Clearly, no simulation can have a shorter thermalisation time.

Marking scheme: *2 points for getting the probability distribution, 8 more for the transition matrix.*

7. A particle on a one-dimensional lattice performs a continuous time random walk with unit transition rates for one or two steps forward and back. No other step lengths are allowed. An ensemble of such systems is started off at lattice site 0 at time $t = 0$. What is the variance of the position of the particles in lattice units at any time after this? (20)

This is a simple generalisation of the continuous-time random walk considered in Lecture 6, and can be solved by the same method. The probability of finding a particle at site n, p_n, follows

$$\frac{dp_n}{dt} = p_{n+2} + p_{n+1} + p_{n-1} + p_{n-2} - 4p_n.$$

This is the only new ingredient. The remainder of the solution follows exactly the same steps as in the lecture notes.

The generating function $F(s,t) = \sum_n s^n p_n(t)$, follows

$$\frac{dF}{dt} = \left(s^2 + s + \frac{1}{s} + \frac{1}{s^2} - 4\right) F,$$

and hence,

$$F(s,t) = \exp\left[t\left(s^2 + s + \frac{1}{s} + \frac{1}{s^2} - 4\right)\right].$$

The moments are given by

$$\left\langle n^i \right\rangle = \left(s\frac{\partial}{\partial s} \right)^2 F(s,t)\Big|_{s=1}$$

This gives

$$\langle n \rangle = t\left(2s^2 + s - \frac{1}{s} - \frac{2}{s^2}\right) F(s,t)\Big|_{s=1} = 0.$$

As a result, the variance is the same as the second moment—

$$\left\langle n^2 \right\rangle = \left[t^2\left(2s^2 + s - \frac{1}{s} - \frac{2}{s^2}\right)^2 + t\left(4s^2 + s + \frac{1}{s} + \frac{4}{s^2}\right)\right] F(s,t)\Big|_{s=1} = 10t.$$

Marking scheme: *3 points for correctly writing the evolution equation, 2 more for using the generating function, 3 more for writing the correct equation for the the generating function, 2 more for solving it, 2 more for writing the formula for the moments, 3 more for the 1st derivative, 5 more for the 2nd derivative.*

8. Solve the one-dimensional three state Potts model by the transfer matrix method, *i.e.*, find the free energy density, $\log Z(\beta)/N$, for the Hamiltonian

$$H = -\sum_{i=1}^{N} \delta_{s_i, s_{i+1}}$$

when $s_i \in \{0,1,2\}$, and periodic boundary conditions are used, but the limit $N \to \infty$ is taken. (20)

Each bond factor is $\exp \beta$ *if the spins at the two ends are equal. If the two spins are unequal, then the bond factor is 1. As a result, the transfer matrix is*

$$T = \begin{pmatrix} w & 1 & 1 \\ 1 & w & 1 \\ 1 & 1 & w \end{pmatrix},$$

where $w = \exp \beta$*. In the limit* $N \to \infty$ *only the largest eigenvalue is needed.*

The largest eigenvalue must have a symmetric eigenvector $(1,1,1)^T$*. Hence the largest eigenvalue,* λ_0*, can be obtained by just multiplying* *the T over this vector. This gives $\lambda_0 = 2 + w$. (A lengthier solution is to find the eigenvalues by solving a cubic.)*

Finally, the free energy density

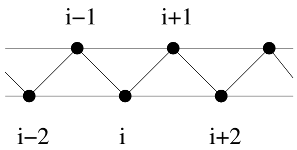
$$F(\beta) = \frac{1}{N}\log Z(\beta) = \log\left(2 + \mathrm{e}^{\beta}\right).$$

Marking scheme: *5 points for writing the correct transfer matrix, 5 points for realising that only the largest eigenvalue is needed, 5 more for finding it, 5 more for the free energy density.*

9. Find the high-temperature expansion of the Ising model on a one-dimensional lattice with nearest and next-to-nearest neighbour bonds—

$$H = -\sum_{i=1}^{N} s_i s_{i+1} - \sum_{i=1}^{N} s_i s_{i+2},$$

when $s_i \in \{-1, 1\}$, and periodic boundary conditions are imposed. Carry out the expansion up to order $\tanh^3\beta$. (20)



A one-dimensional lattice with next-to-nearest neighbour bonds is shown above. The high temperature expansion of the partition function is obtained by the expansion

$$Z = \sum_{\{s\}}\prod_{i=1}^{N}\cosh\beta\,(1 + s_i s_{i+1}\tanh\beta)\cosh\beta\,(1 + s_i s_{i+2}\tanh\beta).$$

The coordination number of each site is 4. Hence the number of bonds on the lattice is $2N$. The leading term of the expansion is

$$Z = \cosh^{2N}\beta + \cdots$$

The higher order terms are powers of $\tanh\beta$. The linear and quadratic terms vanish since they cannot form closed loops.

The smallest closed loop contains 3 bonds. Look at the coordinates of the vertices of such a triangle, and call the smallest one i. Of the three triangles which contain the site i, only one has i as the smallest coordinate. Since each triangle is uniquely labelled by i, the number of such triangles is i. (Alterantively, note that each triangle contains three sites, and there are three triangles at each site. Hence the number of triangles is N.)

Hence the expansion is

$$Z = \cosh^{2N}\beta\left[1 + N\tanh^3\beta + \cdots\right].$$

Marking scheme: *5 points for the leading term, 5 points for finding that the next two terms vanish, 10 points for the cubic term.*