Applying the SIR model for epidemics: a mathematical explainer

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These are notes on the SIR model. I explain the mathematics behind the concepts of social distancing, flattening the curve, and the easing of load on medical services. I also define the standard epidemiological and medical terms called case resolution time (T), case fatality ratio (f) and basic reproduction rate (R_0).
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I. INTRODUCTION

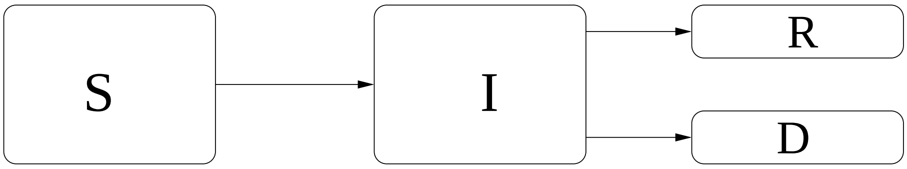

FIG. 1: Epidemiological models envisage individuals moving from one compartment to another as they catch and recover from a disease.

The simplest compartmental model [1] of mathematical epidemiology is discussed here. The population is divided into four compartments— S, which is the susceptible population, I, which is the infected population, R, which is the recovered population, and D, which is the number of deaths (see Figure 1). I will neglect births, under the common assumption that the death rate is larger than the birth rate. The simplest models involve a well-mixed population, *i.e.*, one in which these quantities are dependent only on time and not on the location.

The model is the coupled set of ODEs

$$\dot{S} = -\alpha SI, \qquad \dot{I} = \alpha SI - (\beta+\gamma)I, \qquad \dot{R} = \beta I, \qquad \dot{D} = \gamma I. \tag{1}$$

Note that the equations imply that the total population $P = S + I + R + D$ is conserved (on adding the equations, the right hand sides mutually cancel, yielding zero, and the left hand side becomes $\dot{P}$). This is due to births being neglected. Another crucial assumption is that the recovered persons have immunity. Note that any solution with $I = 0$ leads to R, D, and S constant. More generally R and D are non-decreasing, and S is non-increasing.

Before any further analysis, it is useful to select units appropriate to this problem. This is called dimensional analysis and is a basic tool of the physicist.

Time is the only dimensional quantity in this problem. Note that β and γ have the same dimensions. The dimensionless ratio $f = \gamma/\beta$ is the *case fatality ratio*. The time for cases to resolve into either recovery or death is $T = 1/(\beta+\gamma)$. This is called the *case resolution time*. This is an unit of time natural to the problem. In these time units the equations become

$$\dot{S} = -\frac{\alpha}{\beta+\gamma}\, SI, \qquad \dot{I} = \frac{\alpha}{\beta+\gamma}\, SI - I, \qquad \dot{R} = \frac{1}{1+f}\, I, \qquad \dot{D} = \frac{f}{1+f}\, I. \tag{2}$$

The quantities S, I, R, and D are numbers, and hence dimensionless. However, it is interesting to ask what happens under a scaling $P \to \lambda P$. Multiply all population counts by λ. Clearly, the equations remain unchanged if we also simultaneously scale $\alpha \to \alpha/\lambda$. Any choice of λ is allowed; we may choose in particular $\lambda = 1/P$. This means that the coefficient of the growth term in the equation for I is now $R_0 = \alpha P/(\beta+\gamma)$. This is called the *basic reproduction rate* (see page 7 of [2]). This process of scaling is equivalent to saying that the population is effectively a dimension, and we can use a choice of units to set $P = 1$, and measure α in these units. We now remove the redundant equation and write

$$\dot{S} = -R_0\, SI, \qquad \dot{I} = R_0\, SI - I, \qquad R + D = 1 - S - I, \qquad D/R = f, \tag{3}$$

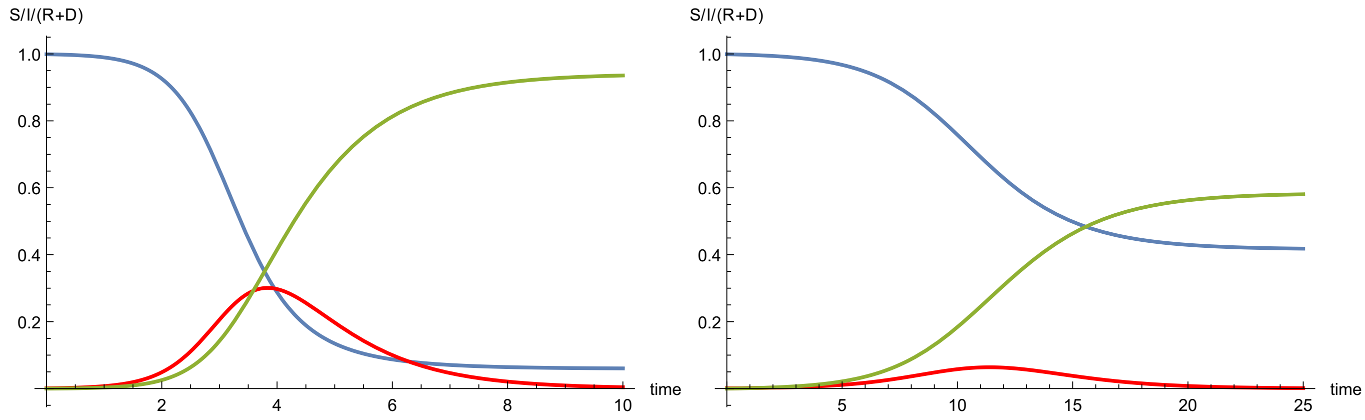

FIG. 2: The solution of the SIR equations in natural units. The curve in blue denotes $S(t)$, that in red $I(t)$, and in green $R(t) + D(t)$. The panel on the left is for $R_0 = 3$, that on the right for $R_0 = 1.5$. With decreasing R_0 the peak in $I(t)$ is lowered and the duration of the epidemic is extended. This has been called **flattening the curve**. Note the dramatic effect on the total cases: for $R_0 = 3$ only 2% of the population remains uninfected, for $R_0 = 1.5$ more than 40% are never infected. Note that time is measured in units of the case resolution time, which for COVID-19 is estimated to be 3 weeks, about the same as for chicken pox.

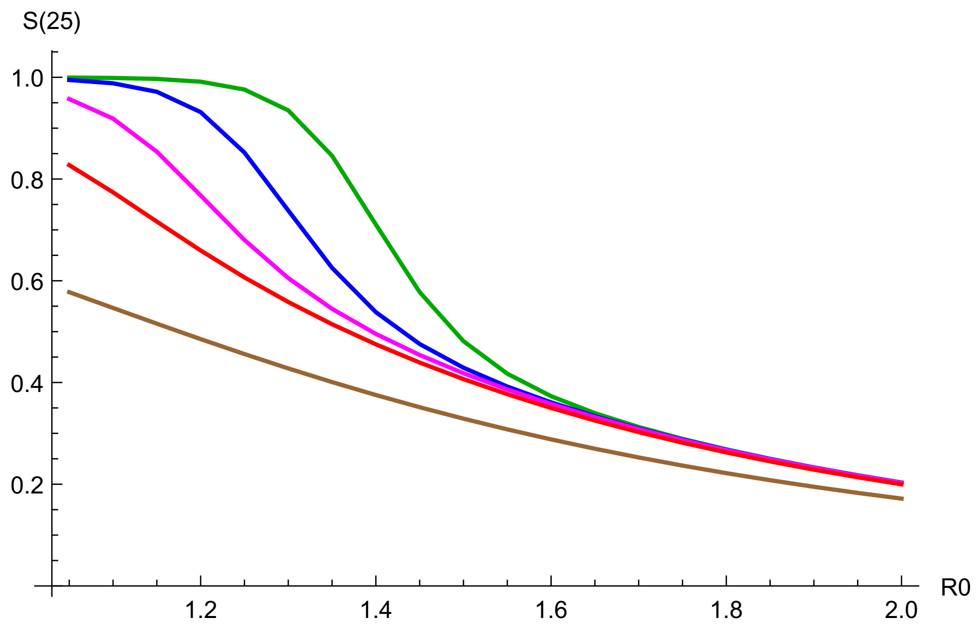

FIG. 3: $S(25)$ is shown as a function of R_0 for different initial conditions $I(0)$, 10^{-5} in green, 10^{-4} in blue, 10^{-3} in gold, 0.01 in red, and 0.1 in brown.

with all quantities S, I, R, and D being less than unity. These are the SIR equations in units natural to the problem. The solutions of the equations in eq. (3) are now applicable to any situation: different diseases differ only by the choice of R_0 and the unit of time, which is the case resolution time. They are applicable for any country, because the solutions are expressed as fraction of the population of the country.

The single parameter in the problem, R_0, depends on two factors: one is the virulence of the pathogen, and the other is the frequency with which susceptible and infected people meet. The simplest epidemiological intervention is to reduce R_0 by decreasing the rate at which susceptible and infected people meet each other. When the infection is apparent, for example, in measles, quarantining infected people is the route taken to do this. For COVID-19, when the infection takes days to manifest itself, **social distancing decreases** R_0.

The general nature of solutions of eq. (3) is easy to understand. I increases when $S > 1/R_0$, and decreases when $S < 1/R_0$. If $R_0 < 1$ then the infection cannot spread and $S \simeq 1$. If $R_0 > 1$, then I increases initially, S decreases, and when $S < 1/R_0$, then the number of infections turns over and begins to decrease. In Figure 2 we compare the solutions for different R_0 of the SIR equations in natural units. When R_0 is decreased the number of people who contract the disease $(R + D)$ drops significantly. However, the duration of the epidemic is also longer. This is now being called **flattening the curve**.

Another way to look at this flattening of the curve is the plot the number of uninfected people, S, at late times as a function of R_0. This is shown in Figure 3. By controlling R_0, the number of healthy people can be incressed. A smaller number of infected people can give limited medical and industrial infrastructure a better chance to cope with an ongoing epidemic. We may call this **easing the load**. Also note that if the load is eased then contact tracing and isolation can be brought into play so that the effect is to decrease $I(0)$, which in turn leads to further decreased load.

This model has been used a lot recently, because it gives a simple and essentially correct overall picture of the progress of a contagion. However, there are many details which may require more complex models. For example, the population may not be homogeneous, and more "compartments" may be needed. Individuals vary a little in how suceptible they are to an infection and how easily their body is able to fight it off. Incorporating this may require stochastic differential equations. Some of these extensions will be dealt with elsewhere.

[1] W. O. Kermack and A. G. McKendrick, *A contribution to the mathematical theory of epidemics*, Proc. Royal Soc. Lond. A 115, 700 (1927).

[2] F. Brauer, P. van den Driessche, and J. Wu (eds), *Lecture Notes in Mathematical Epidemiology* Springer. (Currently free to download)