Plasmas and Dimensional Analysis

Sourendu Gupta

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A plasma is any material which is overall charge neutral but contains mobile charge carriers, and hence conducts current. The simplest plasma that you can make and study is a solution of a salt in water. This is the reason why an electric current flows through a salt solution. The electrons inside a piece of metal also make up a plasma. The material inside a tube light or a compact fluorescent lamp (CFL) is a plasma. The earth's atmosphere contains a plasma— the ionosphere. The hot gases in the sun are a plasma. Much of the tenuous material in interstellar space exists as a plasma. Most matter in the universe was a plasma until the sky became transparent.

1 Microscopic variables and collective effects

What are the microscopic parameters which characterize the plasma? The magnitude of the charge of each mobile element, e, and its mass, m, are two such quantities. The average distance between two charge carriers, λ is a third microscopic parameter. The number density of charge carriers is $n = 1/\lambda^3$. Finally, the average energy of each charge carrier, E, is a microscopic parameter. If the charge carriers in a plasma are in thermal equilibrium with a temperature T, then $E \simeq k_B T$, where k_B is the Boltzmann constant. The notation used here means that E is equal to $k_B T$ times a numerical constant which is within an order of magnitude of 1. We can choose to use units in which the Boltzman constant, $k_B = 1$, so that T has units of energy[1]. So one can take as the microscopic parameters the four quantities, e, m, n and T.

Another choice of units we shall make for convenience is that the the permittivity of free space, $4\pi\epsilon_0 = 1$. As a result, Coulomb's law can be written as

$$\mathbf{F} = \frac{e^2}{r^2}\hat{\mathbf{r}}, \tag{1}$$

where $\mathbf{F}$ is the force between two charges e separated by a distance r and $\hat{\mathbf{r}}$ is the unit vector pointing from one charge to another. Since an unit vector can be taken to be dimensionless, the Coulomb's law gives the dimension of electric charge,

$$[e] = [M]^{1/2}[L]^{3/2}[t]^{-1}. \tag{2}$$

[1]In these units 1 K is 1.38×10^{-23} J, or 8.62×10^{-5} eV.

The Coulomb can then be defined as the charge which exerts a force of 8.988×10^9 N on a like charge placed at a distance of 1 m. The magnitude of an electron's charge is 1.602×10^{-19} C, which is 1.600×10^{-14} $\mathrm{Kg}^{1/2}\mathrm{m}^{3/2}/\mathrm{s}$.

Now, given the microscopic parameters, we can build a single dimensionless quantity out of them—

$$K = \sqrt{\frac{T}{e^2 n^{1/3}}}. \tag{3}$$

It is clear from Table 1 that $K \gg 1$ for many plasmas.

What does this plasma parameter signify? To understand this, let us examine the average distance of closest approach of two like charges in the plasma. The average thermal energy of a particle in the plasma is T. The distance of closest approach to another like charge, d, occurs when the Coulomb energy equals the thermal energy—

$$d = \frac{e^2}{T} = \left(\frac{1}{n^{1/3}}\right)\frac{e^2 n^{1/3}}{T} = \frac{\lambda}{K^2}. \tag{4}$$

If K is large, then d is small. So, on the average, particles are at separations where the Coulomb potential energy can be neglected with respect to the thermal kinetic energy. Such a plasma is called a weakly coupled plasma. If K is of order 1 or smaller, then the Coulomb forces are extremely important. Such plasmas are called strongly coupled plasmas. In the rest of this article we will consider weakly coupled plasmas.

We have identified an extremely small length scale in the problem— λ/K^2. It turns out that there could be a much larger length scale in the problem created by the collective effects of many particles acting together. Such a collective scale could be

$$\lambda_D \simeq K\lambda = \sqrt{\frac{T}{e^2 n}} \tag{5}$$

This emergent scale is called the Debye screening length. The number of particles which lie within a cube of side λ_D is K^3. This number is called the plasma parameter, Λ. This is the number of particles whose collective effects build the Debye screening length. For weakly coupled plasmas, this is a large number.

That argument sounds a little glib; so we examine it from a slightly different angle. We discovered a small dimensionless parameter in the problem, $1/K$. Then we argued that if there is a large length scale, λ_D, in the system then one must have

$$\frac{\lambda}{\lambda_D} = f\left(\frac{1}{K}\right), \tag{6}$$

where f is an unknown function whose form we have to argue about. Such equations are the content of dimensional analysis.

To proceed further, we make an argument in two parts. The first part is that $f(0) = 0$, because the large length scale must emerge out of a full-scale computation of the physics at the microscopic level. In other words, if $f(0) \neq 0$, then it would mean that the ratio on the left hand side is independent of the microscopic parameters contained in Λ— a perfectly unlikely occurrence. The second part of the argument is that if $f(x)$ is a nice and well behaved function at $x = 0$ we should be able to do a Taylor expansion, and by retaining the leading linear term in $1/\Lambda$, obtain the expression in eq. (5).

This part of the argument could sound a little suspect, since there is no dimensional reason why one cannot do a complicated computation and find $f(x) = \exp(-1/x)$. Beyond the argument that $f(0) = 0$, strictly speaking one should resort to the analysis of the dynamics. For a classical plasma, a dynamical analysis involving simple electrostatics gives the result that $f(x) \propto x$. Many computations do give rise to smooth and well behaved functions, so the second part of the argument often holds. It is usually sensible to make such a simple assumption to begin with and check it later through a more thorough analysis. In a complicated problem it is also possible to go the other way— check the result of a very involved computation to see whether or not such a simple idea holds. If it doesn't, then one must trace through the computation and find the reason why.

System	n (m^{-3})	T (° K)	$K = \Lambda^{1/3}$	λ_D (m)
Interstellar gas	10^6	10^4	2321	2.3×10^1
Gaseous nebulae	10^8	10^4	1077	2.3×10^0
Ionosphere	10^{12}	10^3	73	7.3×10^{-3}
Solar (corona)	10^{12}	10^6	2321	2.3×10^{-1}
(atmosphere)	10^{20}	10^4	11	2.3×10^{-6}
(interior)	10^{33}	10^7	2	2.3×10^{-11}
Lab plasma (tenuous)	10^{17}	10^4	34	7.3×10^{-5}
(dense)	10^{22}	10^5	16	7.3×10^{-7}
(thermonuclear)	10^{22}	10^8	500	2.3×10^{-5}
Metal	10^{29}	10^2	0.03	7.3×10^{-12}

Table 1: Characterizing some common plasmas. Here n is the number density of charge carriers, T the temperature, Λ the plasma parameter, λ_D the Debye screening length, ω_p the plasma frequency and the charge of each mobile element has been taken to be equal in magnitude to the electron's charge. Metals and the solar core are not weakly coupled plasmas.

Uptil now we have shown that dimensional analysis of a plasma reveals the existence of two length scales apart from the obvious one, the average interparticle spacing, λ. One of these length scales corresponds to the distance at which the Coulomb potential between two charges becomes comparable to their kinetic energy. If this scale is much shorter than λ, then the other scale is much larger, and is a collective scale called the Debye screening length. We examine the physics of this scale next.

2 The physics of plasmas

We are yet to find an interpretation of the length scale, λ_D, in a plasma, so we look for this now. If the charges in the plasma have set up a potential $\Phi(r)$ and are in equilibrium in this, then the particle densities must be

$$n = \exp\left(-\frac{e_i\Phi}{T}\right). \tag{7}$$

Φ must be roughly constant when averaged over a length scale much larger than λ, because of overall charge neutrality.

Now, if an external test charge density, $\delta\rho_{ext}$, is introduced into the plasma, then the potential is disturbed, becoming $\Phi + \delta\Phi$. This induces a change in the particle densities of $\delta n_i = -ne_i\delta\Phi/T$. The net effective charge density is the sum of the external density and that induced by this change in particle number—

$$\delta\rho = \delta\rho_{ext} - \frac{2e^2 n_0 \delta\Phi}{T} = \delta\rho_{ext} - \frac{\delta\Phi}{\lambda_D^2} \tag{8}$$

Thus, the externally introduced charge density polarizes the medium by separating out the charges by a small amount.[2]

If we make a self-consistent determination of the change in the potential using Laplace's equation, $\nabla^2\delta\Phi = \delta\rho$, then we find

$$\left(\nabla^2 - \frac{1}{\lambda_D^2}\right)\delta\Phi = \delta\rho_{ext}. \tag{9}$$

Assume that the external test charge was a point charge of magnitude q, then the solution to the Laplace equation can easily be found by standard methods. This solution is

$$\delta\Phi(r) = \frac{q(r)}{r}, \qquad \text{where} \qquad q(r) = q\mathrm{e}^{-r/\lambda_D}. \tag{10}$$

The charges in the plasma are polarized a little by the external charge so as to shield it out.

Of course the total charge of a system must still be conserved. If we take a volume of a plasma in some container and put a test charge inside it, then oen can do experiments and find the total charge inside the container. By conservation of charge, this must equal the test charge that one introduced. Since that charge is screened, by attracting a net positive charge towards it from the plasma, there must be an induced negative charge on the walls of the container, which precisely balances it.

The phenomenon of Debye screening can be said to renormalize charge in a plasma. A charge, q, is measured by the Coulomb force, F, it produces on an unit positive charge placed at some distance r. In a vacuum, this yields a value

[2] We have absorbed the dimensionless number $\sqrt{2}$ into the definition of λ_D, since we are not forbidden to do so by dimensional analysis.

of the charge that is independent of distance. However, due to the mechanism of screening in a plasma, such a measurement would yield a charge, $q(r)$, which is distance dependent, and becomes negligibly small when $r \gg \lambda_D$.

What about the response of a plasma to time varying fields? There is a microscopic time scale in the problem. Since, a typical microscopic velocity is $v = \sqrt{T/m}$, the average time taken by a typical charge carrier to meet a neighbour is

$$\tau = \lambda/v = m^{1/2} n^{-1/3} T^{-1/2} \tag{11}$$

The frequency of close collisions is $1/\tau$. If we consider a light wave travelling through a plasma, it is clear that the electric and magnetic fields of the wave will accelerate the charge carriers. If the frequency, ω, of light is sufficiently high ($\omega\tau \gg 1$) then the charge carriers will describe a complicated orbit with amplitude much less than λ. As a result, a charge carrier accelerated by the wave will seldom interact with another, and there are unlikely to be collective effects. At some threshold, ω_p, the external field accelerates a charge carrier in one direction for long enough that it can move by a distance greater than λ, and hence be very likely to collide with other charged particles, thereby losing the energy it gained from the wave. This would lead to dissipation, and a consequent attenuation of the wave. One expects this to happen for

$$\omega_p \simeq \frac{K}{\tau} = \sqrt{\frac{e^2 n}{m}}. \tag{12}$$

Note that ω_p is proportional to e.

Now, turning this whole argument backwards, eq. (12) can be written in the form

$$K = \tau\omega_p = (n\lambda_D^3)^{1/3} = \Lambda^{1/3}. \tag{13}$$

Hence, a large value of the plasma parameter, Λ implies that there are many particles inside one Debye volume in the plasma, or that in time $1/\omega_p$ each charge carrier collides many times with others. Thus, by coarse graining plasmas over distances larger than λ_D one can give an effective description of the plasma which no longer contains the microscopic physics parameters.

This is really quite remarkable. To appreciate this, recall that we work with such coarse grained description of matter in many different contexts—electrodynamics of continuous media, fluid dynamics, elastic theories of solids, particle physics. In many systems the microscopic physics is not so clearly amenable to analysis that yields length and time scales for the coarse graining into continuum description of matter. The simplicity of Coulomb interactions allows us to perform just such an analysis in the case of a plasma.[3]

3 Quantum mechanics and relativity

It is interesting to continue this analysis into other regimes, for example into plasmas where quantum effects may be important, such as the conduction elec-

[3] A general framework for deriving coarse grained physics is called the renormalization group.

trons in a metal. Then Planck's constant introduces another dimensional quantity $\hbar$ into the analysis. In the classical plasma there was only one dimensionless number Λ. Now there is another—

$$\eta = \frac{\hbar n^{1/3}}{(mT)^{1/2}} = \frac{\hbar/\lambda}{\sqrt{mT}}. \tag{14}$$

The last expression above is in the form of the ratio of two lengths. The mean thermal momentum of the gas, $\sqrt{mT}$, defines a "thermal de Broglie wavelength" $\lambda_T = \hbar/\sqrt{mT}$. η is the ratio of λ_T to the average interparticle spacing, λ. Thus, eq. (6) is replaced by

$$\frac{\lambda}{\lambda_D} = f_q\left(\frac{1}{\Lambda}, \eta\right). \tag{15}$$

If T is so large that many such thermal wavelengths fit into λ then $\eta \ll 1$. This is equivalent to taking the classical limit, $\hbar \to 0$ (note the simplicity of analysis that flows from using dimensionless variables). In this limit we recover eq. (6) through the argument that $f_q(1/\Lambda, 0) = f(1/\Lambda)$.

When the temperature is so small that quantum effects are very large, (*i.e.*, $\eta \gg 1$) it is more useful to neglect T and combine the variables together to form the parameter

$$\Lambda_q = \left(\frac{e}{\hbar}\right)\sqrt{\frac{m}{n^{1/3}}} = \left(\frac{e}{\hbar}\right)\sqrt{m\lambda} \tag{16}$$

which can be used to write a new scaling function in the small parameters $1/\Lambda_q$ and $1/\eta$. The case when η is neither too large nor too small is much more complicated, and the full dynamical computation must be resorted to.

What about a classical relativistic plasma? Perhaps such materials could be found in astrophysical objects— inside stars or supernovæ? The speed of light, c, is a new variable in the problem. This allows us to construct a second dimensionless quantity,

$$\gamma = \sqrt{\frac{T}{mc^2}} \tag{17}$$

which is the usual relativistic factor for time dilation or Lorentz contraction. This is the ratio of the thermal energy and the rest energy of the charge carrier. Clearly, this is negligible if the temperature of the plasma is small. Then the analysis reduces again to the case of the classical plasma. For an electron $mc^2 \approx 0.5$ MeV, so T has to be about 10^{10} K for relativistic effects to become important.[4]

Instead of analyzing this situation in classical relativity, it is more useful to analyze it for a relativistic quantum field theory. The reason is that when the thermal energies of the charge carriers are much larger than their rest energies, collisions between two charge carriers can create more of them. To deal with such particle creation accurately, one must take into account the proper quantum

[4]Temperatures in a supernova explosion are expected to be in the range of 10^8 K. Plasmas in this situation cannot be considered relativistic.

nature of the particles. In this case one has three dimensionless variables. One can choose these to be Λ, η and γ as before.

However, it is better to recognize that the electric charge can be written in a dimensionless unit called the fine structure constant

$$\alpha = \frac{e^2}{4\pi\hbar c}\,. \tag{18}$$

If the charge carriers have charge equal in magnitude to the electron's, then $\alpha \approx 1/137$. Although this is a small number, α cannot be neglected, since then we would lose the electromagnetic interactions which are the cause of collective effects. The second dimensionless variable is the analogue of the classical plasma parameter is

$$K = \sqrt{\frac{T}{\hbar c n^{1/3}}} = \sqrt{\frac{\lambda T}{\hbar c}}\,. \tag{19}$$

The scaling law then becomes

$$\frac{\lambda}{\lambda_D} = f_{QFT}\left(\alpha, \frac{1}{K}, \gamma\right), \tag{20}$$

and we seem to have a rather complicated three variable problem at hand.

However, there is an enormous simplification in the ultra-relativistic limit, $\gamma \to \infty$. In this limit we can neglect the rest energy in comparison to the thermal energy, and simply set the limit $m \to 0$. But then the number of particles in a quantum gas need not be fixed. A well-known example is the photon gas which makes up black-body radiation. In such a gas the number density, n, must simply be replaced by the entropy density, s. Now for black-body radiation one knows that the energy density, $\epsilon \propto T^4$, a fact which goes by the name of the Stefan-Boltzmann law. The pressure in such a gas is also proportional to T^4, hence, using the thermodynamic identity $Ts = \epsilon + P$, one finds that $s \propto T^3$. Since this argument is purely dimensional, it must also hold for any massless quantum gas. Then, from eq. (19) one finds that Λ becomes independent of the microscopic quantities. As a result, the scaling formula in eq. (20) simplifies to

$$T\lambda_D = 1/f_{QFT}(\alpha). \tag{21}$$

All these results are most easily obtained by using the so-called natural units. Since c is an universal constant, one recognizes that L and t are not independent variables. A natural choice of units is $c = 1$ and hence $[L] = [t]$. Next, since $\hbar$ is an universal constant, one recognizes that setting $\hbar = 1$ creates a simple system of units where $[L] = 1/[M]$. In these natural units $e = \sqrt{4\pi\alpha}$ is dimensionless and n is clearly proportional to T^3. If $m \to 0$ then the result in eq. (21) follows quickly. This equation is more often written in the form

$$m_D = f_{QFT}(\alpha)T, \tag{22}$$

where $m_D = 1/\lambda_D$ (in natural units) is called the Debye mass.

Although it has nothing to do with plasmas, it is interesting to carry the construction of natural units further and throw in Newton's universal constant of gravity, G. Then setting G to unity gets rid of all dimensional quantities from physics. One can get the dimensionless values of any physical quantity by using the two following basic units—

$$M_P = \sqrt{\frac{\hbar c}{G}} \qquad \text{and} \qquad L_P = \sqrt{\frac{\hbar G}{c^3}}\,. \tag{23}$$

M_P in usual units has dimensions of mass and is called the Planck mass. L_P is called the Planck length and has units of length in the usual units. The corresponding time unit is L_P/c.

4 Non-Abelian plasmas

Debye screening also occurs in plasmas made of elementary particles which have interactions other than the electromagnetic. Such ultra-relativistic quantum plasmas are relevant to the study of early universe cosmology. The relevant forces are the strong interactions of quarks and gluons and the weak interactions of all particles. These resemble the electromagnetic interactions in that they are all gauge theories and are characterized by a dimensionless coupling constant analogous to α. The distinction between these forces is that they have non-Abelian gauge groups, unlike electromagnetism, which has an Abelian gauge group. The only relevance that this esoteric fact has to the program of dimensional analysis is that the strong coupling α_s can be large. As a result, the scaling function in eq. (21) cannot be appproximated by its linear term. The following results therefore go beyong dimensional analysis, and required detailed techniques of field theory.

The first attempts to study this function beyond the leading term proceeded by making a Taylor expansion of $f(\alpha_s)$, known as the perturbative expansion. It was soon realized that a Taylor expansion of this function cannot be carrried out beyond the fourth order (this is called the Linde problem). A few years later it was realized that there is not even any Taylor expansion in α_s because after the linear term one comes across a term in $\alpha_s^{3/2}$ (this is the result of an analysis called the Braaten-Pisarski resummation). The status of this problem is that one now knows that

$$f(\alpha_s) = a\alpha_s + b\alpha_s^{3/2} + c\alpha_s^2 + d\alpha_s^2 \log\alpha_s + \cdots, \tag{24}$$

where the coefficients a and b can be determined by simply assuming that α_s is small, but the coefficients c and d require a full solution of the problem. This uses a numerically intensive method called lattice gauge theory, which is nowadays carried out on supercomputers. These non-Abelian plasmas are probably being created daily at a laboratory near New York city called the Brookhaven National Laboratory.