# Chemical potential without sweat

**Sourendu Gupta, TIFR, Mumbai**

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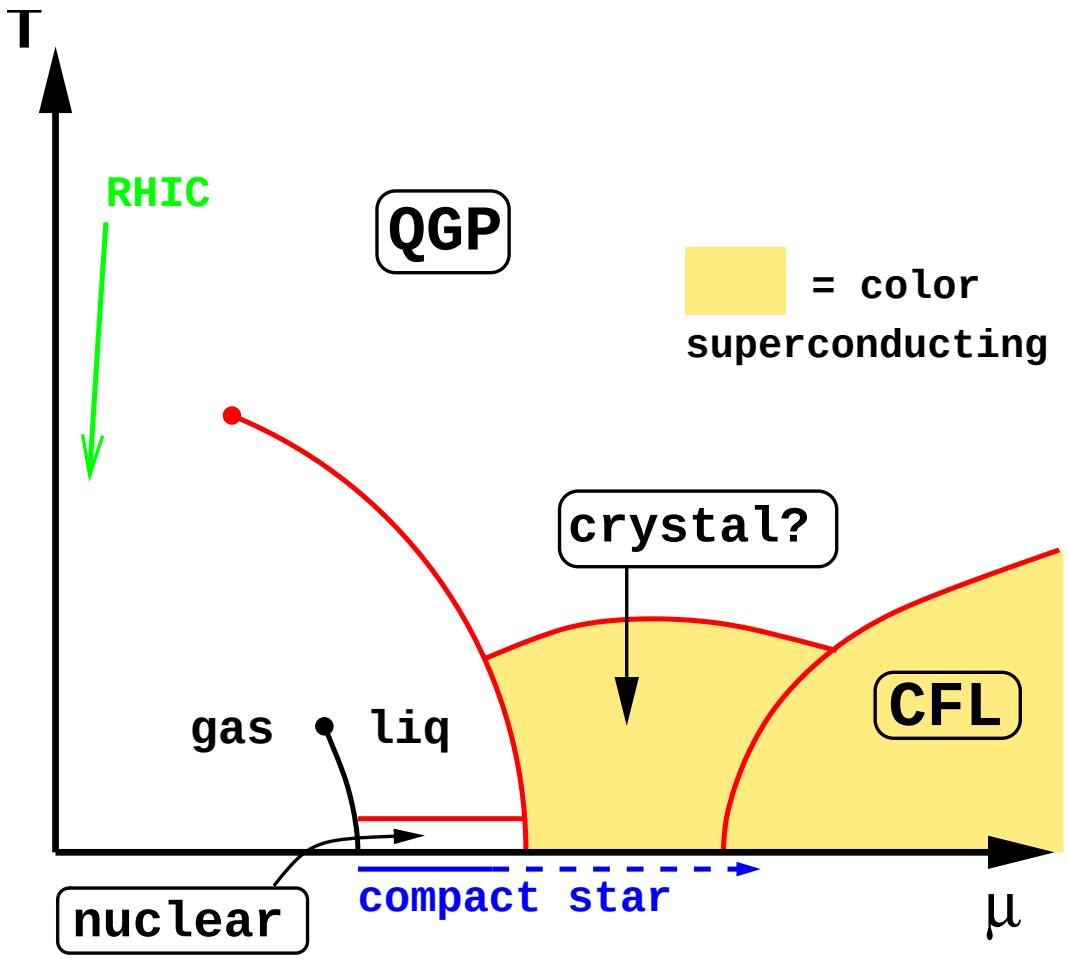
1. Why chemical potential? Why sweat?
2. Two possibilities: reweighting and Taylor series expansion.
3. Quark number susceptibilities
4. Non-linear number susceptibilities
5. Main results

# Why chemical potential?



# Why sweat?

$$Z = \mathrm{e}^{-F/T} = \int DU \mathrm{e}^{-S} \prod_f \det M(U, m_f, \mu_f)$$

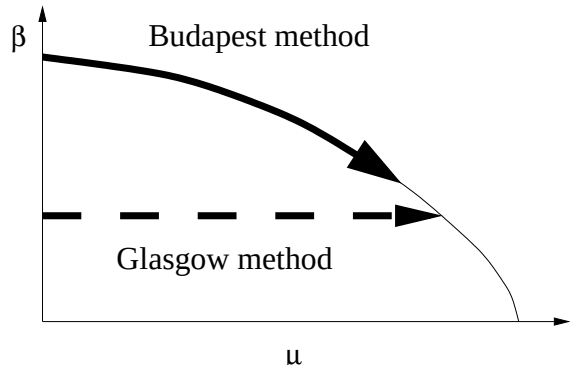
- If $M^\dagger = Q^\dagger M Q$ for some $Q$, then clearly $\det M$ is real.
- For $\mu = 0$ $Q = \gamma_5$. For $\mu \neq 0$ no $Q$ exists.
- Monte Carlo simulations of $Z$ fail.
- However, $Z$ remains real and non-negative: thermodynamics is safe.

All lattice computations done with $m_u = m_d$ ($N_f = 2$). Some also with $m_s/T_c \approx 1$ ($N_f = 2 + 1$). Many with $\det M = 1$ ($N_f = 0$).

# Reweighting

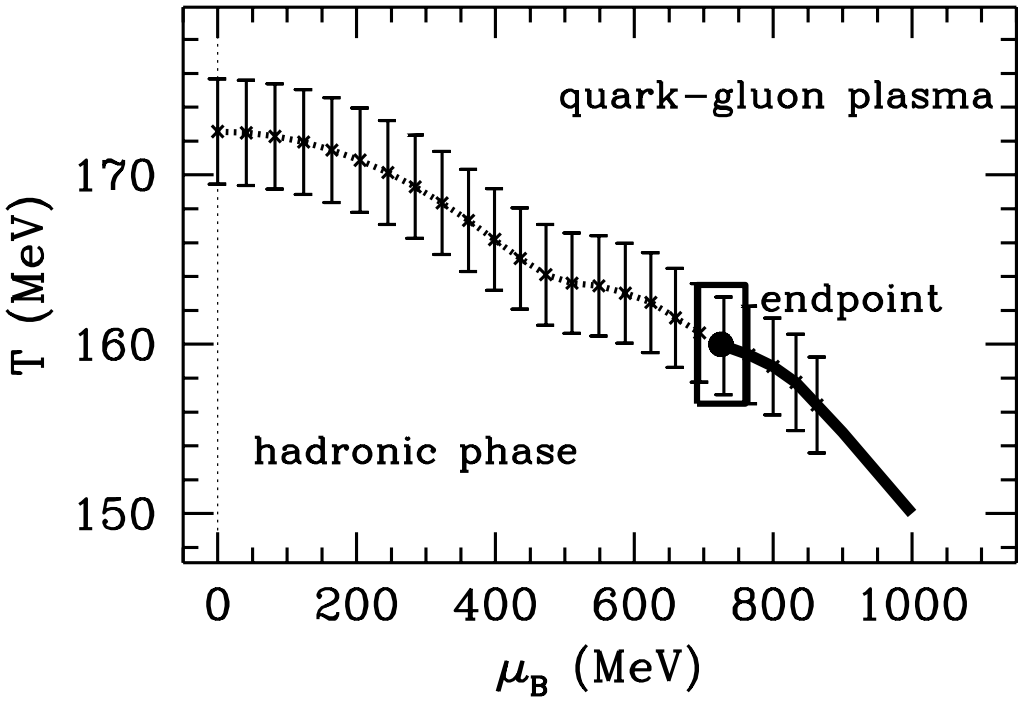
Do simulations at $\mu_f = 0$, re-express expectation values in terms of these—

$$\langle O \rangle_\mu = \frac{\langle O \exp(-\Delta\mathcal{S}) \rangle}{\langle \exp(-\Delta\mathcal{S}) \rangle} \quad \text{where} \quad \mathcal{S} = S - \sum_f \mathrm{Tr}\ \log M_f,$$



# Reweighting: results

Reweighting done for coarse lattices ($N_t = 4$) and $N_f = 4$, 2 and 2+1.




Z. Fodor and S. D. Katz, *J. H. E. P.*, 03 (2002) 014.

# Reweighting: variants

- Express the reweighting in terms of derivatives of Z with respect to chemical potential.
  C. R. Allton *et al.*, *Phys. Rev.*, D 66 (2002) 074507

- Simulate imaginary chemical potential (positive $\det M$) and do analytic continuation. This actually the same as above.
  M. D'Elia and M.-P. Lombardo, hep-lat/0209146
  P. De Forcrand and O. Philipsen, *Nucl. Phys.*, B642 (2002) 290

# Taylor Expansion

$$\frac{1}{V}F(T,\mu_u,\mu_d) = \frac{1}{V}F(T,0,0) + \sum_f n_f\mu_f + \frac{1}{2!}\sum_{fg}\chi_{fg}\mu_f\mu_g + \cdots$$

where the quark number densities and susceptibilities are—

$$\begin{aligned}
n_f &= \frac{T}{V}\frac{\partial \log Z}{\partial \mu_f}\bigg|_{\mu_f=0} \\
\chi_{fg} &= \frac{T}{V}\frac{\partial^2 \log Z}{\partial \mu_f \partial \mu_g}\bigg|_{\mu_f=\mu_g=0} \\
\chi_{fgh\cdots} &= \frac{T}{V}\frac{\partial^n \log Z}{\partial \mu_f \partial \mu_g \partial \mu_h \cdots}\bigg|_{\mu_f=\mu_g=\cdots=0}
\end{aligned}$$

# Derivatives

Derivatives of $\log Z$ can be expressed in terms of derivatives of $Z$. The latter can be constructed by the chain rule.

$$Z_f = \frac{\partial Z}{\partial \mu_f} = \int DU \mathrm{e}^{-\mathcal{S}} \mathrm{Tr}\ M_f^{-1} M_f'.$$

Next,

$$\begin{aligned} Z_{fg} &= \frac{\partial^2 Z}{\partial \mu_f \partial \mu_g} = \int DU \mathrm{e}^{-\mathcal{S}} \Big\{ \mathrm{Tr}\ M_f^{-1} M_f' \mathrm{Tr}\ M_g^{-1} M_g' \\ &\quad + \delta_{fg} \left( \mathrm{Tr}\ M_f^{-1} M_f'' - \mathrm{Tr}\ M_f^{-1} M_f' M_f^{-1} M_f' \right) \Big\}. \end{aligned}$$

S. Gottlieb *et al.*, *Phys. Rev. Lett.*, 59 (1987) 2247

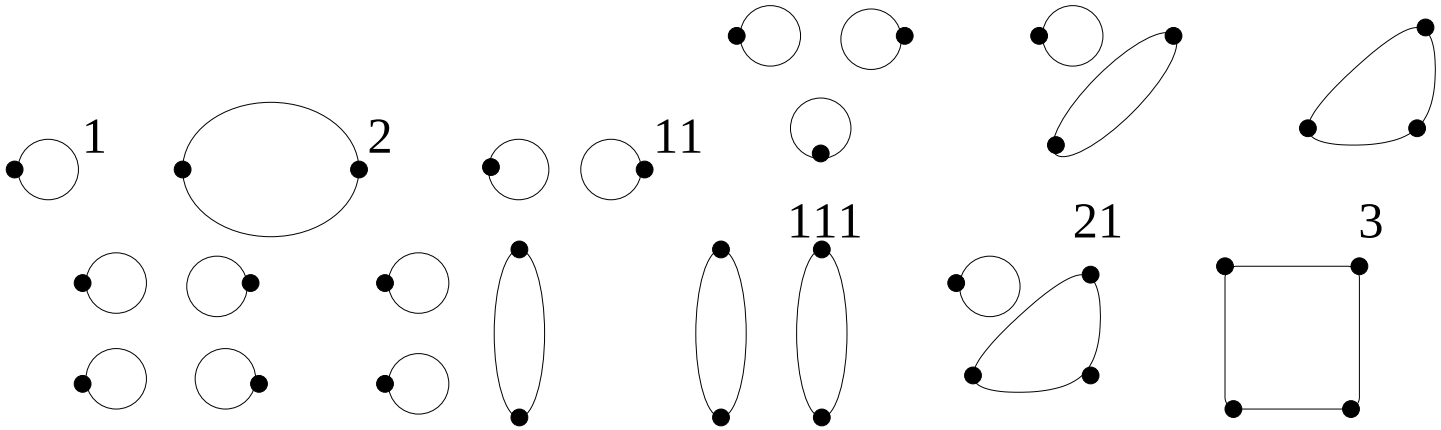
Odd derivatives vanish for $\mu_f = 0$ by CP symmetry.

# Diagrammatic representation

Some definitions:

$$Z_f = Z\langle \mathcal{O}_1 \rangle, \quad \mathcal{O}_{n+1} = \frac{\partial \mathcal{O}_n}{\partial \mu_f}, \quad \mathcal{O}_{ij\ldots} = \mathcal{O}_i \mathcal{O}_j \cdots$$

Diagrams:




S. Gupta, *Acta Phys. Pol.*, B 33 (2002) 4259

# Quark number susceptibilities

- Fluctuations of conserved quantities in heavy-ion collisions are related to $\chi_{uu}$. Isospin fluctuations are related to $\chi_3 = \chi_{uu} - \chi_{ud}$, charge fluctuations can also be constructed out of these. M. Asakawa *et al.*, *Phys. Rev. Lett.*, 85 (2000) 2072; S. Jeon and V. Koch, *ibid.*, 85 (2000) 2076

- Under certain conditions strangeness production rate can be related to the strange susceptibility, $\chi_{ss}$. R. V. Gavai *et al.*, *Phys. Rev.*, D 65 (2002) 054506

- The pressure at finite chemical potential is essentially determined by the susceptibility.

- $\chi_3$ is the zero momentum Euclidean finite temperature vector propagator and hence closely related to a transport coefficient— the DC electrical conductivity of quark matter.

# Some notation

With two degenerate flavours of quarks, in flavour space the linear susceptibilities form the matrix

$$\begin{pmatrix} \chi_u & \chi_{ud} \\ \chi_{ud} & \chi_u \end{pmatrix}$$

Redefining $\mu_0 = \mu_u + \mu_d$ and $\mu_3 = \mu_u - \mu_d$, this matrix becomes

$$\begin{pmatrix} \chi_u + \chi_{ud} & 0 \\ 0 & \chi_u - \chi_{ud} \end{pmatrix}$$

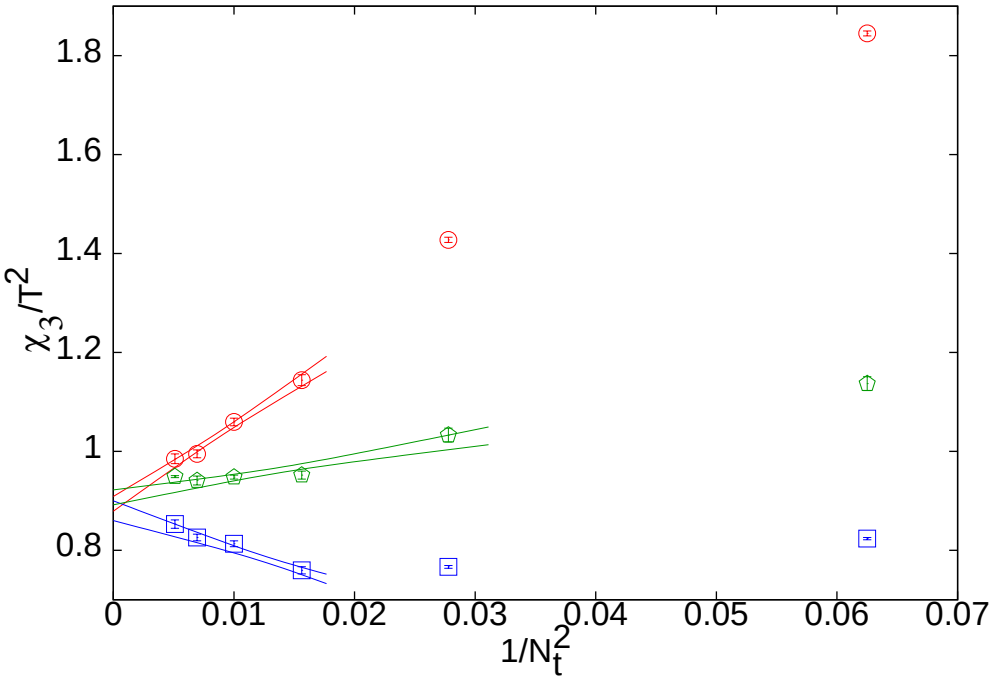
We define

$$\begin{aligned} \chi_3 &= \chi_u - \chi_{ud} = \left\langle \mathrm{Tr}\ M^{-1}M'M^{-1}M' - \mathrm{Tr}\ M^{-1}M'' \right\rangle \\ \chi_{ud} &= \left\langle \left(\mathrm{Tr}\ M^{-1}M'\right)^2 \right\rangle \quad \text{and} \quad \chi_0 = \chi_3 + 2\chi_{ud} \end{aligned}$$

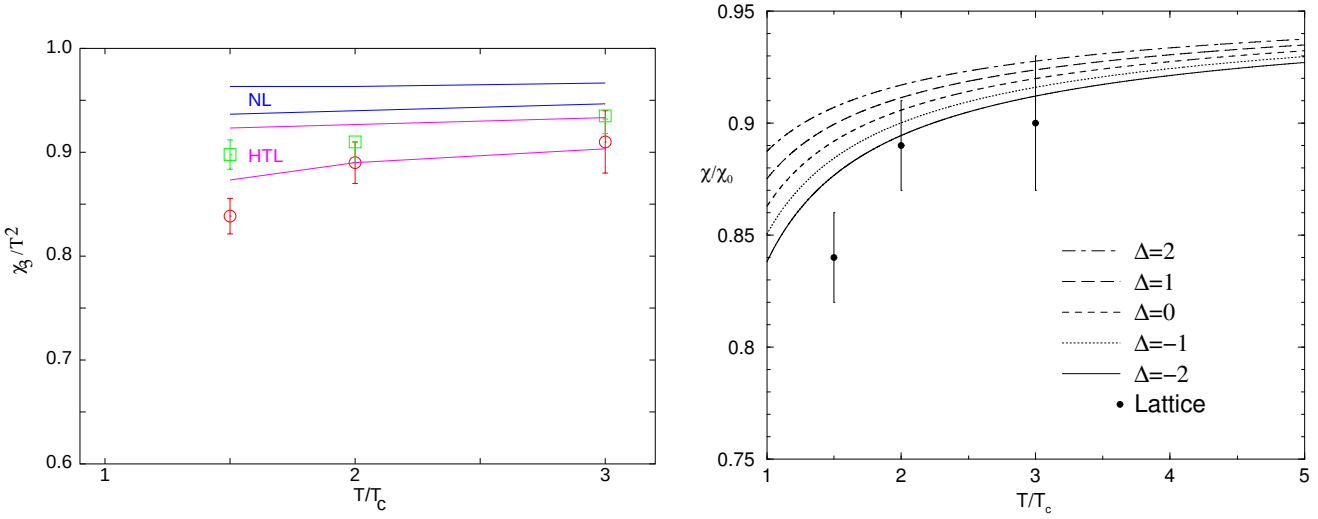
# Finding the continuum limit

Main technical problem is to control the extrapolation to zero lattice spacing. For this we use two different kinds of Fermions (staggered and Naik) and perform simultaneous extrapolation with both: in the quenched theory.

R. V. Gavai and S. Gupta, *Phys. Rev.* D 67 (2003) 034501



# Other methods



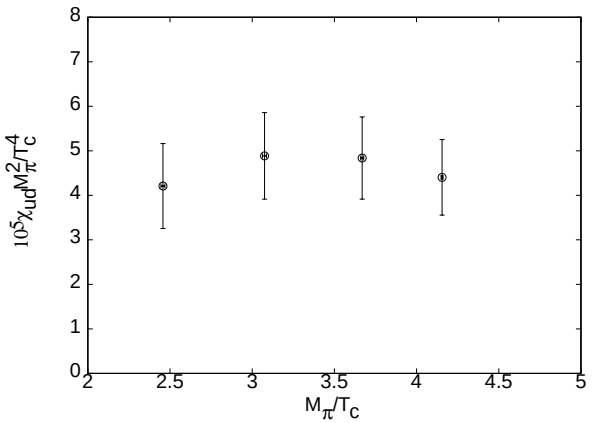

J.P. Blaizot, E. Iancu and A. Rebhan, *Phys. Lett.*, B 523 (2001) 143

A. Vuorinen, hep-ph/0212283

# $\chi_{ud}$

$$\begin{aligned} \mathrm{Tr}\ A &= \frac{1}{2}\overline{r^\dagger A r} \simeq \frac{1}{2N_v}\sum_{i=1}^{N_v} r_i^\dagger A r_i \\ (\mathrm{Tr}\ A)^2 &= \frac{1}{4}\overline{r^\dagger A r\, s^\dagger A s} \end{aligned}$$

$\chi_{ud} = 0$ for all $T > T_c$, but not $T < T_c$



# Prescription

Chemical potential on the lattice is prescription dependent. Prescription involves a factor $f(\mu a)$ for forward propagation of quark by interval $a$ in time, $g(\mu a)$ for anti-quark. Conditions—

1. $f(0) = g(0) = 1$.

2. $f(x) = g(-x)$.

3. $f'(0) = 1$.

4. $f(x)g(x) = 1$. (Hence $f''(0) = 1$)

R. V. Gavai, *Phys. Rev.*, D 32 (1985) 519

# Escape route

$$\begin{aligned}
\mathcal{O}_3 &= \mathrm{Tr}\left[M^{(3)}M^{-1} + 2(M'M^{-1})^3 - 3M''M^{-1}M'M^{-1}\right] \\
\mathcal{O}_4 &= \mathrm{Tr}\left[M^{(4)}M^{-1} + \cdots\right]
\end{aligned}$$

$M^{(n)} = f^{(n)}(0)a^{n-2}M'$ (for $n = 3, 5, \cdots$) and $M^{(n)} = f^{(n)}(0)a^{n-2}M''$ (for $n = 4, 6, \cdots$).

- In HK prescription: $f(x) = \exp x$, and all $f^{(n)}(0) = 1$.
- In BG prescription: $f(x) = (1-x)/\sqrt{1-x^2}$, hence $f^{(3)}(0) = 3$, $f^{(4)}(0) = 9$.

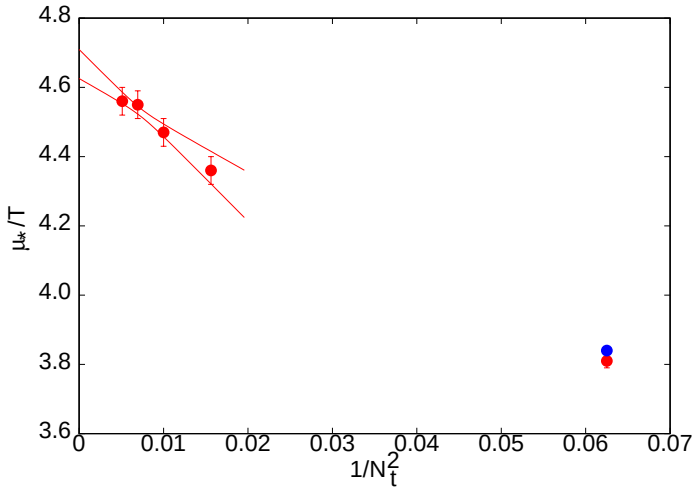
Taylor series expansion is prescription dependent beyond 2nd order at every finite $a$, but prescription independent for $a \to 0$. R. V. Gavai and S. Gupta, hep-lat/0303013

# Taylor series for pressure

$$
\begin{aligned}
P(T,\mu) = F/V &= P(T,0) + \chi_3(T)\mu^2 + \frac{1}{12}\chi_{uuuu}(T)\mu^4 + \mathcal{O}\left(\mu^4\right) \\
&= P(T,0) + \chi_3\mu^2\left[1 + \left(\frac{\mu}{\mu_*}\right)^2 + \mathcal{O}\left(\frac{\mu^4}{\mu_*^4}\right)\right].
\end{aligned}
$$

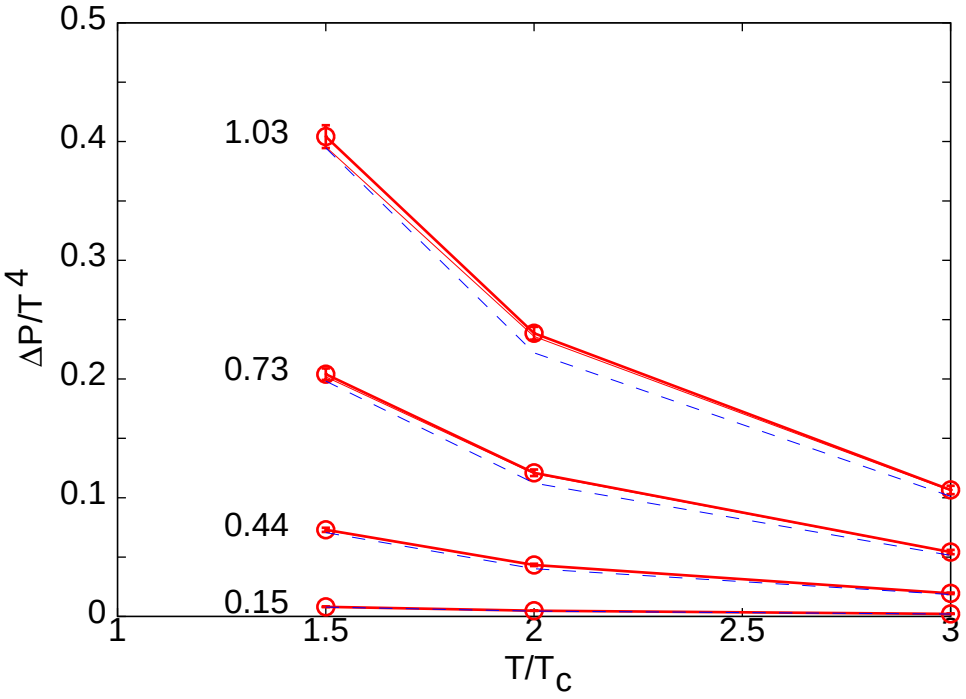
where $\mu_* = \sqrt{12\chi_3/\chi_{uuuu}}$ and other 2nd and 4th order terms have been neglected. Well-behaved for $\mu \ll \mu_*$ if all the higher order terms are small enough. All results can be obtained in the continuum. Term by term improvement of the series is possible. Series should fail to converge near a critical point. Series extrapolation methods should then be used to locate the critical point nearest to $\mu = 0$.

# Radius of convergence



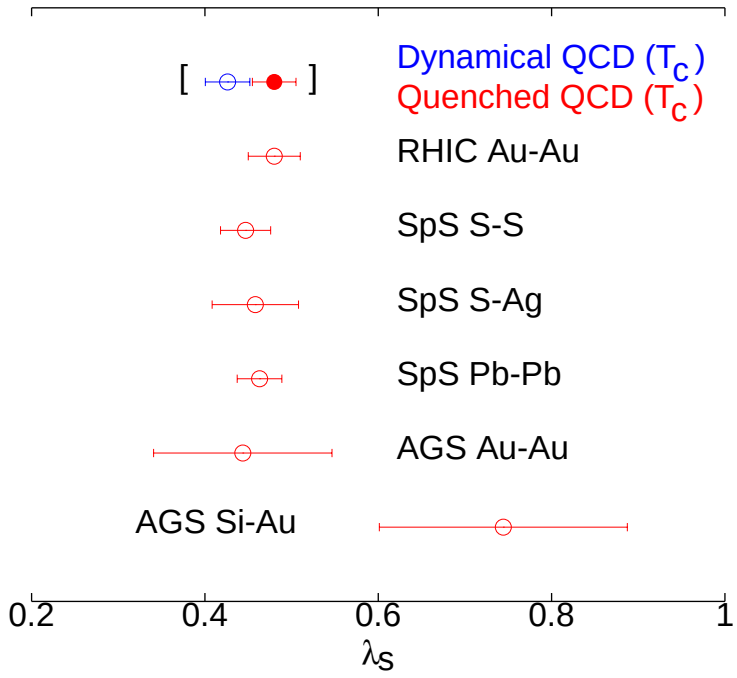

4-th order estimate of $\mu_*$ at $T = 1.5T_c$. At finite $N_t$, the series is insensitive to prescription when $\mu \ll \mu_*$. In the continuum $\mu_*$ is the first estimate of the radius of convergence of the series.

# The pressure




$$\Delta P(T) = P(T,\mu) - P(T,0)$$

# Strangeness production




$$\lambda_s = \frac{\langle n_s \rangle}{\langle n_u + n_d \rangle}$$

# Summary of Results

- Susceptibilities provide a systematic and easy way of computing quantities non-perturbatively at finite chemical potential in the continuum.
- Computation of several high order susceptibilities may allow estimation of the critical end point by series extrapolation methods.
- Fluctuations and strangeness production rate in heavy-ion collisions are related to susceptibilities.
- Susceptibilities allow extension of the equation of state to finite chemical potential.