Transport and fluctuations in finite temperature QCD

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1. (Very brief) introduction to linear response theory, its application to photon production and electrical conductivity.
2. Extraction of spectral function from lattice data using Bayesian methods. Using lattice data to find the time scale of transport phenomena.
3. Fluctuations in equilibrium: specific heats, compressibility, speed of sound, coefficient of expansion.
4. Unbounded fluctuations pinpoint the critical point in the phase diagram of QCD.

Flucutations and dissipation

- The spectrum of fluctuations is encoded in a spectral function. This is central to many different aspects of physics in and near thermal equilibrium.

- They are related to the correlation function (Wiener-Khinchine theorem). I will discuss the extraction of spectral functions from measurements of correlation functions on the lattice.

- They are related to linear response coefficients (Fluctuation-dissipation theorem). I will discuss the extraction of a sample dissipative coefficient— the electrical conductivity.

- They can be measured in experiments. The electrical conductivity is related to a limiting case of photon production. Other fluctuations can be measured directly.

Linear Response Theory

The response, $\mathbf{A}(t)$, of a system to a force $\mathbf{F}(t)$ if non-linear terms are neglected—

$$\mathbf{A}(t) = \int_{-\infty}^{\infty} dt' \chi(t-t')\mathbf{F}(t') \qquad \text{hence} \qquad \mathbf{A}(\omega) = \chi(\omega)\mathbf{F}(\omega).$$

Causality implies $\chi(t) = 0$ for $t < 0$. As a result $\chi(\omega)$ is regular in the upper half plane and dispersion relations follow. The spectral density is the imaginary part of $\chi(\omega)$ as ω approaches the real axis from above. A microscopic computation explicitly relates $\chi(\omega)$ to the retarded propagator. From this follow the Kubo formulæ relating the transport coefficient and the zero energy limit of the spectral density—

$$\chi \propto \lim_{\epsilon \to 0} \int d^3x' \int_{-\infty}^{t} dt'' \mathrm{e}^{\epsilon(t''-t)} \int_{-\infty}^{t''} dt' \langle \mathbf{A}(\mathbf{x}, t)\mathbf{A}(\mathbf{x}', t')\rangle.$$

J. Hilgevoord, *Dispersion Relations and Causal Description*, North-Holland, 1960

Temporal correlators:
electrical conductivity and photon emissivity

The differential photon emissivity is given by—

$$\omega\frac{d\Omega}{d^3p} = \frac{C_{EM}}{8\pi^3} n_B(\omega;T)\rho^\mu_\mu(\omega,\mathbf{p};T) \qquad \text{where} \qquad C_{EM} = 4\pi\alpha\sum_f e_f^2 \approx \frac{1}{21}.$$

In terms of the DC electrical conductivity ($\mathbf{j} = \sigma\mathbf{E}$)

$$\sigma(T) = \frac{C_{EM}}{6}\left.\frac{\partial}{\partial\omega}\rho^i_i(\omega,\mathbf{0};T)\right|_{\omega=0}, \qquad \frac{8\pi^3\omega}{C_{EM}T^2}\frac{d\Omega}{d^3p} = 6\frac{\sigma}{T}.$$

Since $k^\mu\rho_{\mu\nu} = 0$, we have $\rho_{00} = 0$ along the line $\mathbf{p} = 0$. Formally,

$$\rho_{00}(\omega,\mathbf{0};T) = 2\pi\chi_Q\omega\delta(\omega),$$

where χ_Q is the charge susceptibility.

Free field theory

Here by free field theory we will mean an ideal gas of charged fermions. Since they are ideal, all interparticle interactions have been switched off. Their charge only responds to an external field. There is conduction without dissipation, *i.e.*, Ohm's law does not hold.

In such a theory the spectral function at large energy is

$$\rho_{ij}(\omega, \mathbf{0}) \propto \delta_{ij}\omega^2$$

and

$$\rho_{00}(\omega, \mathbf{0}; T) = 2\pi\chi_Q\omega\delta(\omega)$$

where χ_Q is given by its ideal gas value.

Since there is no dissipation, it also follows that this system does not emit photons.

Lattice Correlators

In the (Euclidean) lattice theory one constructs equilibrium correlation functions which are related to the spectral function by—

$$G(\tau, \mathbf{p}; T) = \int_0^\infty \frac{\omega}{2\pi} K(\omega, \tau; T) \rho(\omega, \mathbf{p}; T).$$

In a lattice theory there are N_t points in the τ direction, but there is a continuous infinity of ω.

Replace integral by sum, the linear relation above becomes a set of linear equations: more variables than equations. Inverse of K is ill defined. Convert to a minimisation/Bayesian problem.

Another case where the inverse matrix is ill-defined is when there are more equations than unknowns. In this case the usual method of solution is by least squares.

Regularisation

When the number of variables is larger than the number of equations, maximize the Bayesian probability—

$$P(\rho|G) \propto P(G|\rho)P(\rho) = \exp[-F(\rho)],$$

$$F(\rho) = (G - K\rho)^T \Sigma^{-1} (G - K\rho) + \beta U(\rho)$$

β is a regularisation parameter, Σ is the covariance matrix of the measured G, and $U(\rho)$ is a function which we are free to choose. This function encodes our prior knowledge of the system.

A. N. Tikhonov and V. Y. Arsenin, *Solutions of Ill-posed Problems*, Wiley, New York (1977)

1. Specifying a regulator function is not, in principle, different from parametrising $\rho(\omega)$. In practise, the results may differ.

2. When the errors are large then Σ^{-1} is small and the prior assumptions effect the solution strongly. When the errors are small then Σ^{-1} is large and improper assumptions can sometimes be identified and consequently removed.

Flavours of regularisation

1. Maximum Entropy Method has $U = \sum \rho \log(\rho/\rho_0) - \rho$, where ρ_0 is a free further choice.
   Y. Nakahara, M. Asakawa and T. Hatsuda, *Phys. Rev.*, D 60 (1999) 091503, QCD TARO, *Nucl. Phys.* B (Proc. Suppl.) 63 (1998) 460

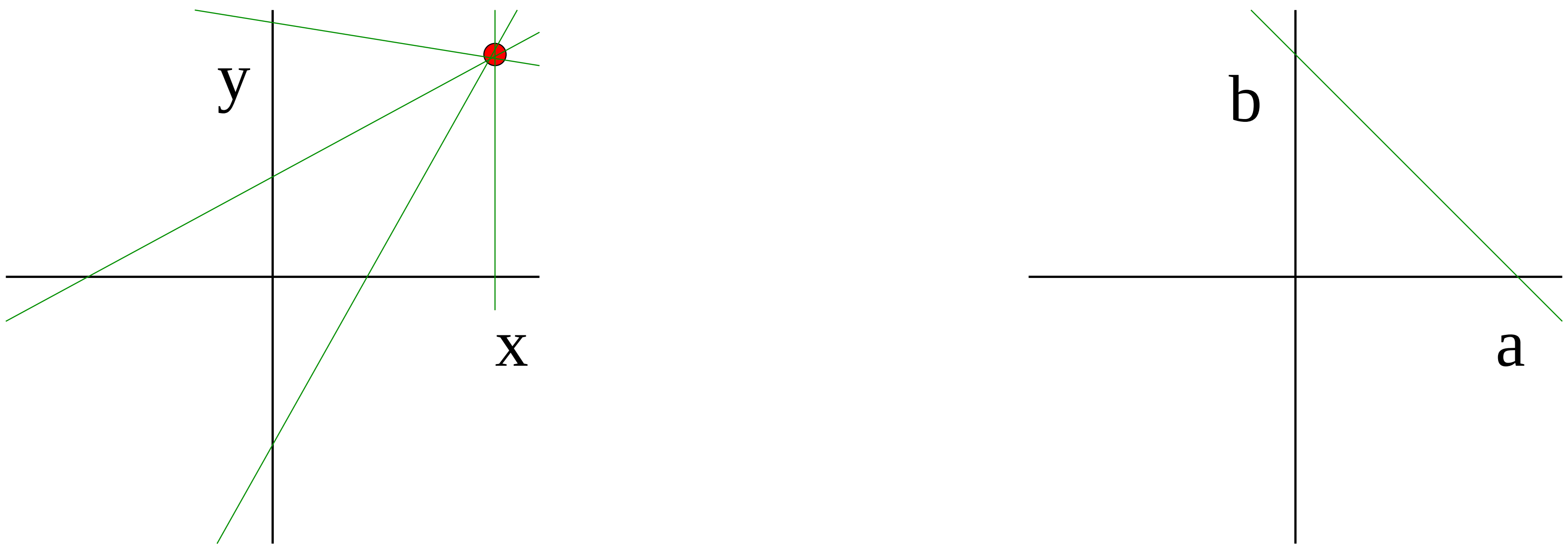
2. A linear regulator is of the form $U = \rho^T L^T L \rho$, where the matrix $L = 1$, D, D^2, *etc.*.
   SG, hep-lat/0301006

3. Include known information into the Bayesian probability.
   G. P. Lepage *et al.*, *Nucl. Phys.*, B (Proc. Suppl.) 106 (2002) 12.

4. Fit a form with small parameters to the functional form, and accept or reject this hypothesis by the usual means.
   Hammurabi?

An example

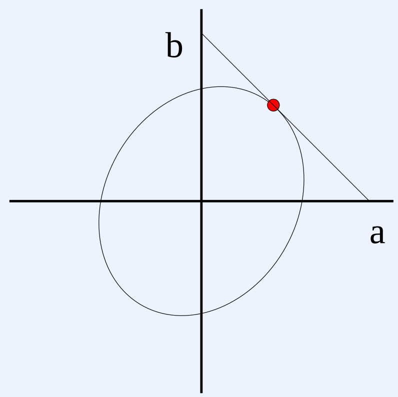
Determine the parameters of the line $y = a + bx$ passing through (1,1)



Simplified version of the actual problem to be solved: $2 \times L^3$ lattice.

Solution: method 1

Method 1: General linear regulator $U = \rho^T L^T L \rho$



For all L the best fit passes through the data.

$$F(a,b) = (1-a-b)^2 + \beta(l_{11}a^2 + l_{22}b^2 + 2l_{12}ab)$$

U is positive definite. The minimum occurs at

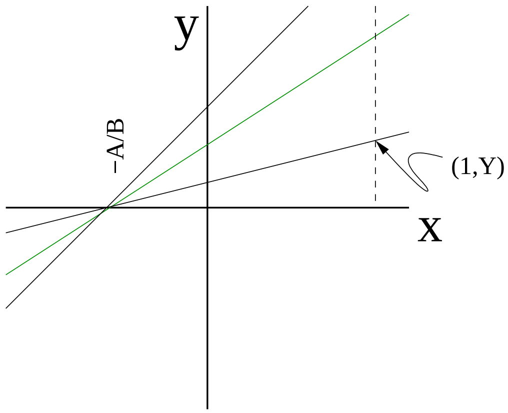
$$M \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{where} \quad M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \beta \begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix}$$

$$\text{Most probable } \beta = 0: \quad \begin{pmatrix} a \\ b \end{pmatrix} = \overbrace{\frac{1}{1+x}\begin{pmatrix} x \\ 1 \end{pmatrix}}^{l_{11}\neq 0\ (x=l_{22}/l_{11})} \quad \text{or} \quad \overbrace{\frac{1}{1+x}\begin{pmatrix} 1 \\ x \end{pmatrix}}^{l_{22}\neq 0\ (x=l_{11}/l_{22})} \qquad (l_{12}=0).$$

Solution: method 2

Method 2: MEM

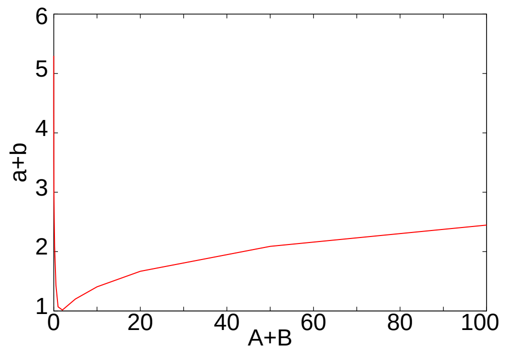
$$F(a,b) = (1-a-b)^2 + \beta\left(a\log\frac{a}{A} + b\log\frac{b}{B} - a - b\right)$$



The minimum is at

$$\frac{a}{A} = \frac{b}{B} = u \quad \text{where} \quad 1 - Yu = \frac{\beta}{2}\log u,$$

and $Y = A + B$. Solutions exist only for $Y > 0$.
If $Y < 1$ then $u > 1$ and vice versa.



The best fit does not pass through the data except when $A + B = 1$

Solution: method 3

Method 3: Partial knowledge

Prior: the line passes through the origin. We can choose the Bayesian probability distribution to be a Gaussian of width σ around the origin—

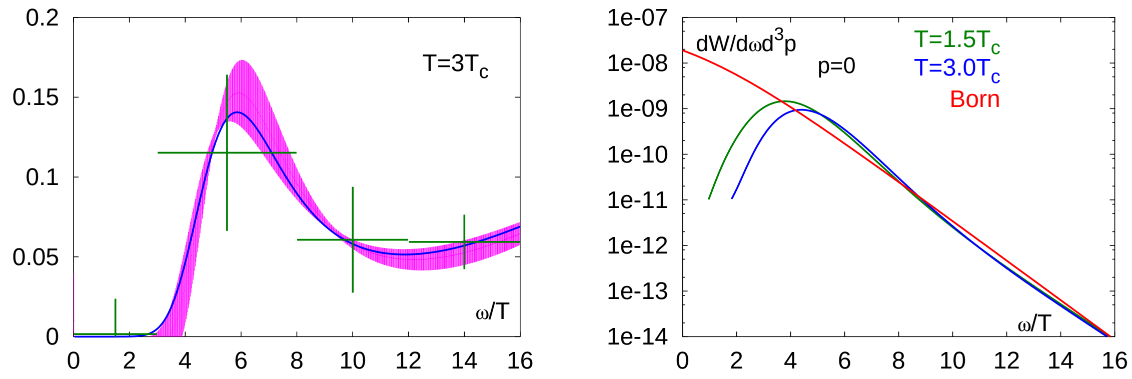
$$F = (1 - a - b)^2 + \frac{a^2}{2\sigma} \quad \text{giving} \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If only information on b is needed then a can be integrated out of the Bayesian probability to give the marginal distribution

$$P(b)db \propto db \exp\left[-(1 - b)^2\right].$$

In this case marginalisation gives b=1 which is the same result as minimisation. This is true if the distribution is unimodal.

Large ω using MEM



F. Karsch et al, Phys.Lett.B530:147,2002— Wilson quarks

Full agreement with Born for $\omega/T \geq 4$.

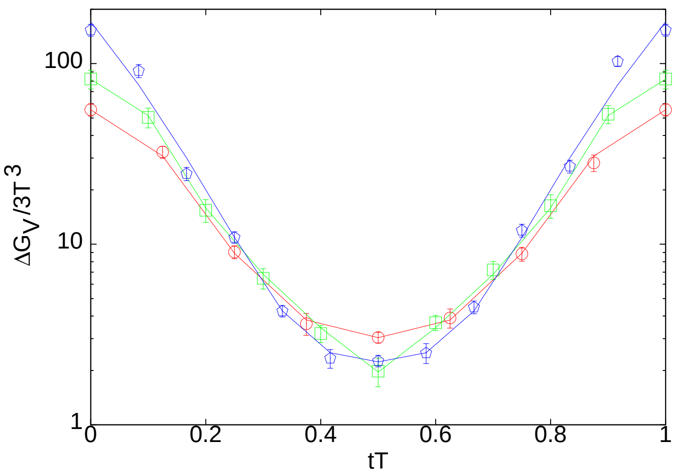
Default model: ideal gas behaviour. Output: $\rho(\omega)$ grows as ω^2 at large ω. Extracted value vanishes as $\omega \to 0$. Need to examine low ω region by another method in more detail.

Small ω using linear regulator

Since the problem is linear, work with

$$\Delta G(\omega, \mathbf{p}; T) = G_{full}(\omega, \mathbf{p}; T) - G_{ideal}(\omega, \mathbf{p}; T).$$

This gets rid of the ω^2 divergence at infinity, at the cost of the positivity of $\Delta\rho$. Use a linear regulator. This shows a bump at small ω. Second and higher bump at ω/T ≈8–9.



SG, hep-lat/0301006

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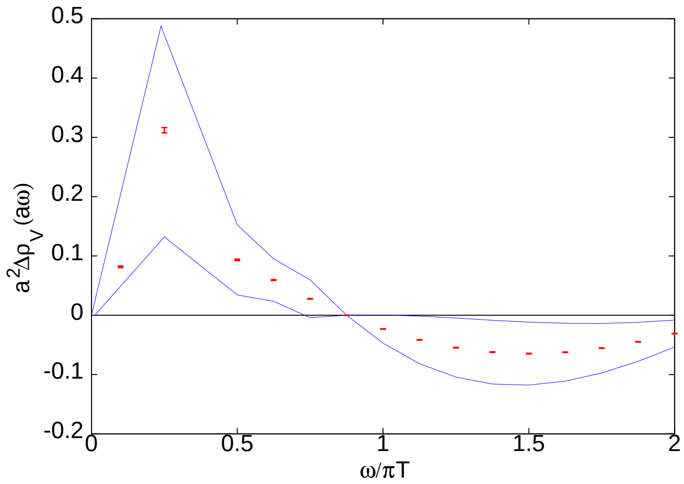

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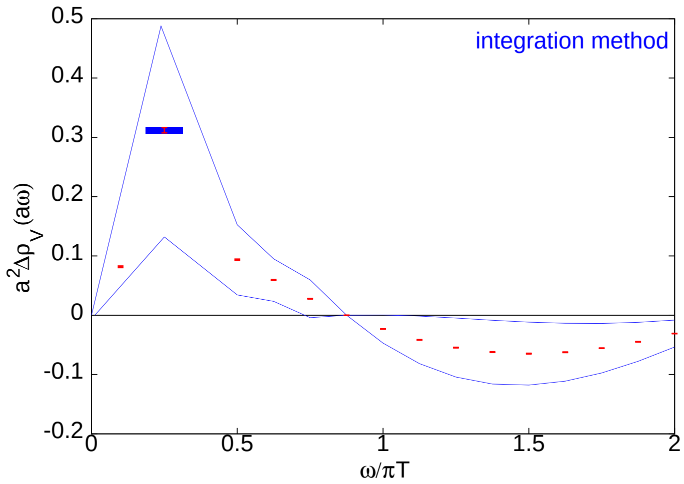

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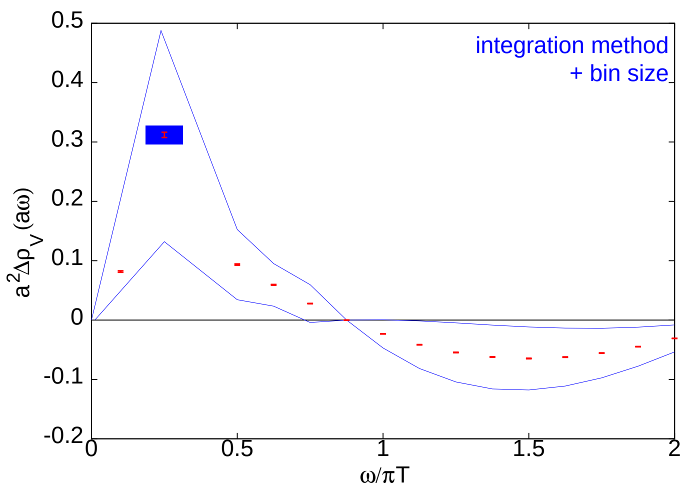

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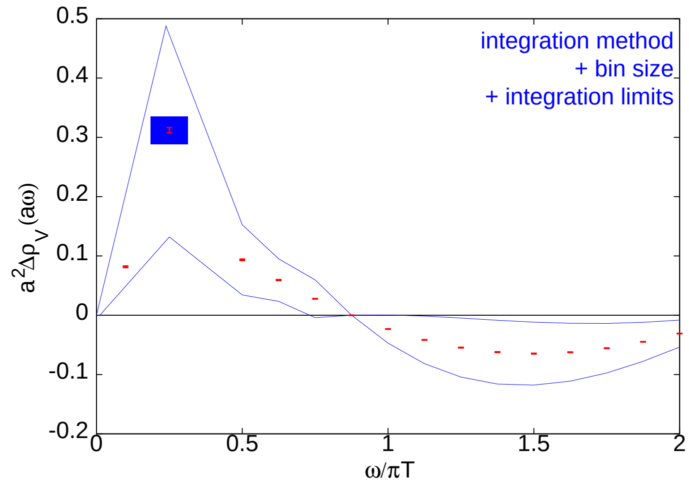

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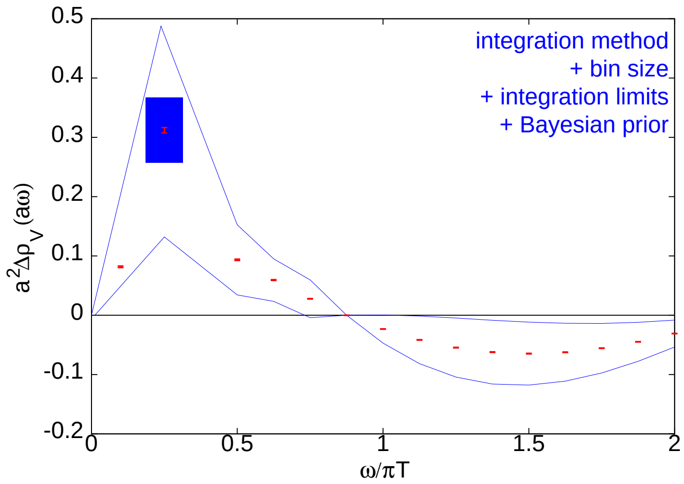

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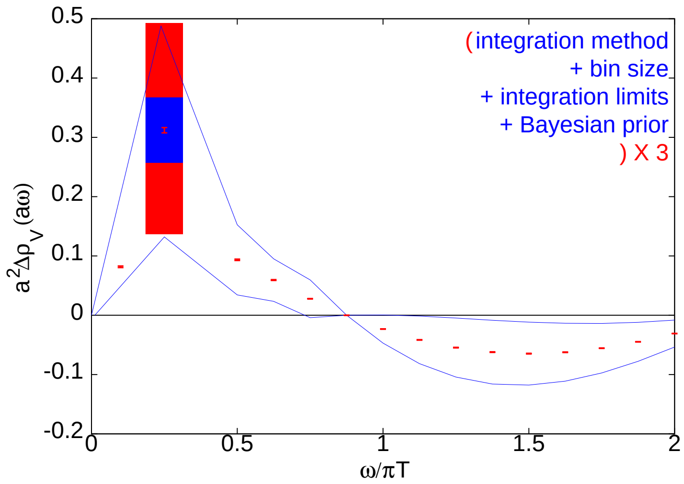


SG, hep-lat/0301006

Lattice gauge theory with parametrised Bayesian methods

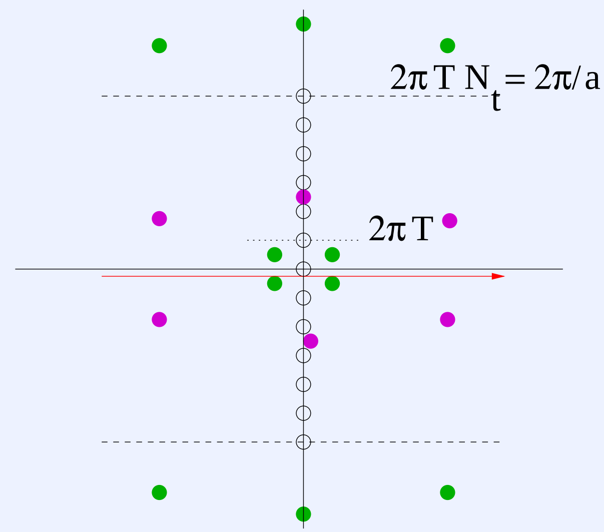
Use a sequence of parametrisations for the spectral density

$$\frac{\Delta\rho}{T^2} = \frac{z\sum_{n=0}^{N}\gamma_n z^{2n}}{1+\sum_{m=1}^{M}\delta_m z^{2m}}.$$

Use with Fourier space correlators—

$$\Delta G(\omega_n, \mathbf{p}; T) = \oint \frac{d\omega}{2i\pi}\,\frac{\Delta\rho(\omega, \mathbf{p}; T)}{\omega-\omega_n}$$

where $\omega_n = 2i\pi nT$.



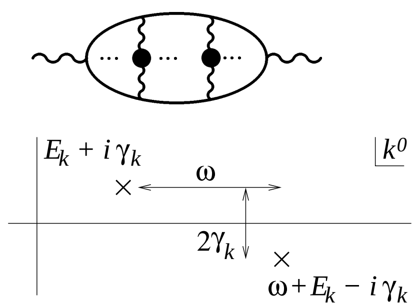
Two types of poles: S_2 is "relaxation time" and S_4 is "transport". S_2 ruled out.

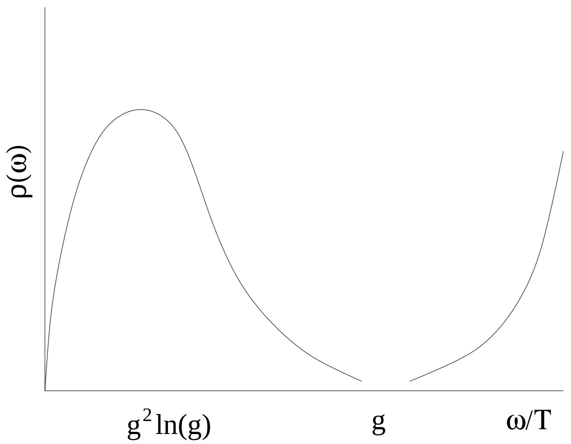
Use χ^2 parameter fitting if $N+M+1 \le N_t$, Bayesian otherwise.

F. Karsch and H. W. Wyld, *Phys. Rev.*, D 35 (1987) 2518; S. Sakai *et al.*, hep-lat/9810031

Pinch singularities and transport

There are pinch singularities at small external energy, ω, from ladder diagrams. These ladder diagrams correspond to multiple scatterings off particles in the plasma.

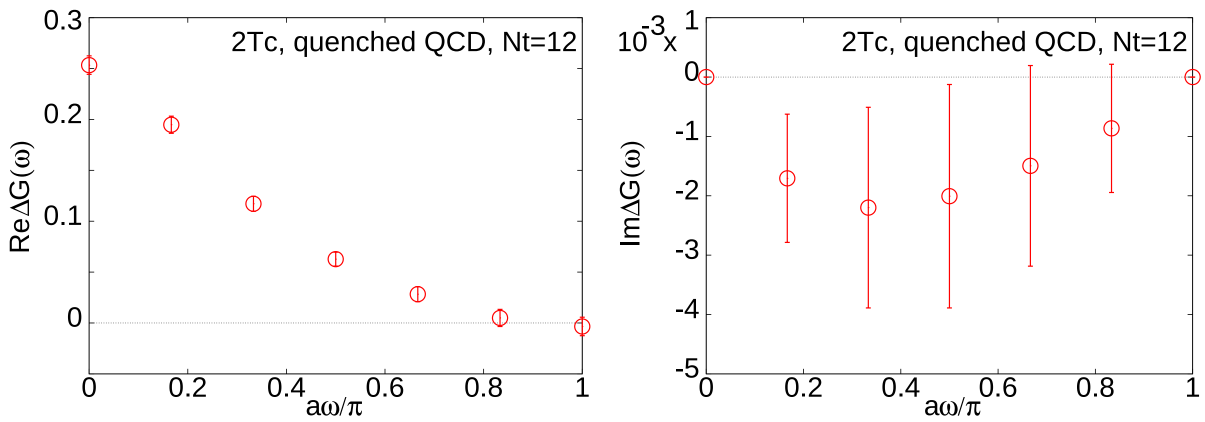




Transport: Arnold, Moore and Yaffe,

G. Aarts and J.M.M. Resco JHEP 0204:053,2002

Lattice results

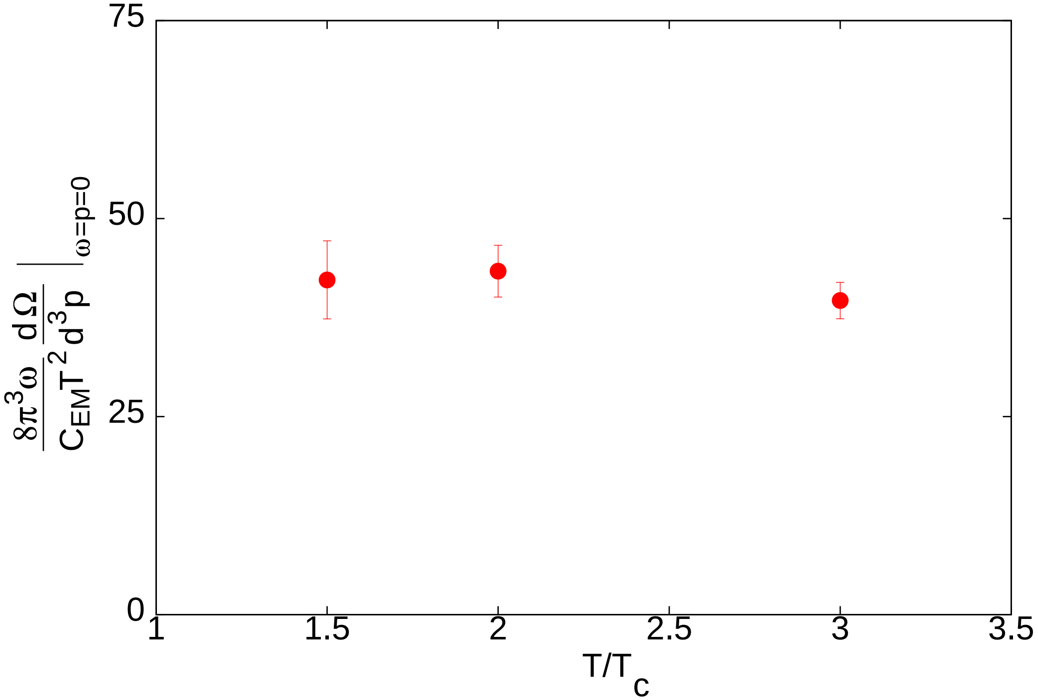


- For $N_t = 12$ fits possible with $|n| \leq 3$.
- Smaller lattices give qualitative results but a fit is not possible.
- S_4 poles at $\omega/2\pi T \approx 0.15$ and $\phi \approx 0.2$.
- AdS/CFT for $N = 4$ SUSY YM gives $\omega/2\pi T = 1$ and $\phi = \pi/4$. Dam Son and A. Starinets.

Electrical conductivity: continuum limit

Electrical conductivity depends only on the parameter γ. Obtain this by marginalising over the remaining parameters. SG, hep-lat/0301006

$\frac{\sigma}{T} \approx 7C_{EM}$ for $1.5 \leq T/T_c \leq 3$



Dynamical scales and phenomenology

1. A soft photon ($\omega \approx T$) emitted in the plasma is reabsorbed if its path length is

$$\ell = \frac{1}{\sigma} \approx \frac{1}{7 C_{EM} T} \approx 3 \text{ fm}.$$

   Typical fireball dimensions at RHIC are a few fm, so the fireball is transparent to soft photons ($\omega \approx 200 MeV$).

2. Typical hadronic length/time scales in the plasma are

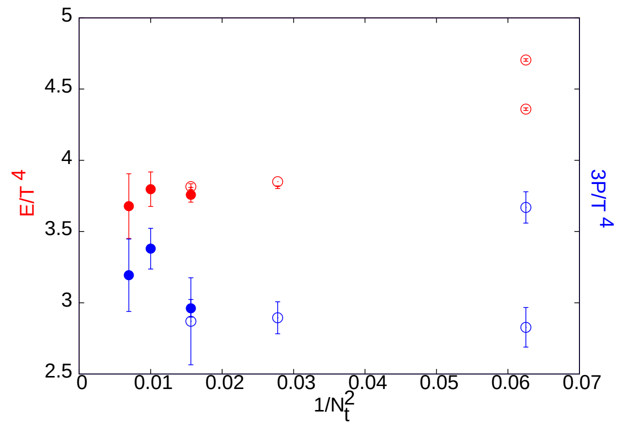
$$\tau \approx \frac{1}{7T} \approx 0.15 \text{ fm}.$$

   Hydrodynamic description of the final state in the plasma works if its thermalisation time is less than 1 fm. Therefore hydrodynamics may work at both RHIC and LHC.

Fluctuations in equilibrium

Lattice computations determine the partition function Z in an ensemble with fixed spatial volume V and temperature T. The EOS is obtained from the two first derivatives—

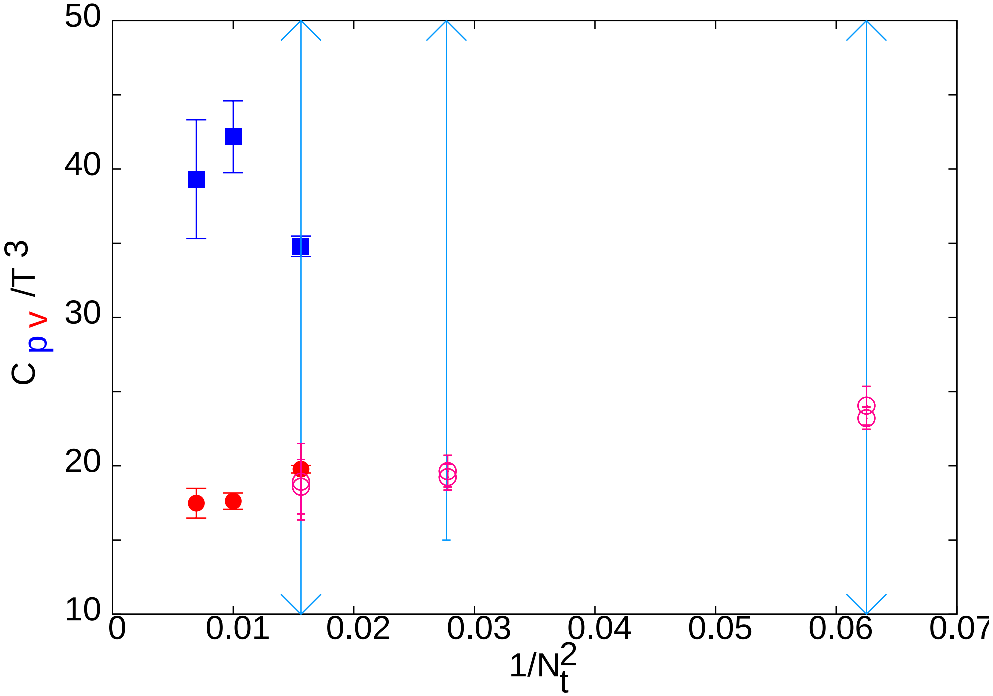
$$E = \frac{1}{V} \left. \frac{\partial \log Z}{\partial T} \right|_V \qquad P = \left. \frac{\partial \log Z}{\partial V} \right|_T .$$



All measures of thermodynamic fluctuations can be built out of the three second derivatives—

$$c_V = \frac{1}{V} \frac{\partial^2 \log Z}{\partial T^2}, \quad 1/\kappa = -V \frac{\partial^2 \log Z}{\partial V^2}, \quad \text{and} \ \frac{\partial^2 \log Z}{\partial V \partial T}.$$

Specific heats

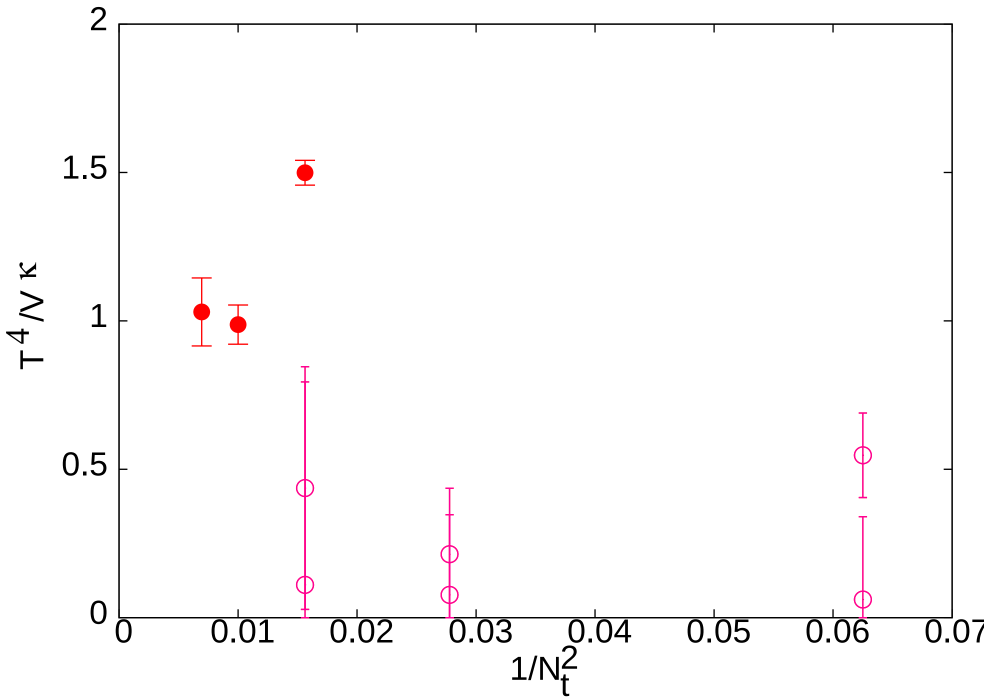


$$c_P \equiv \frac{1}{V}\frac{\partial H}{\partial T} = c_V + T\alpha^2/\kappa, \quad \text{where} \quad \alpha = \frac{1}{V}\frac{\partial V}{\partial T}.$$

First observation of a non-trivial value of c_P. Note that $\gamma = c_P/c_V \approx 2$.

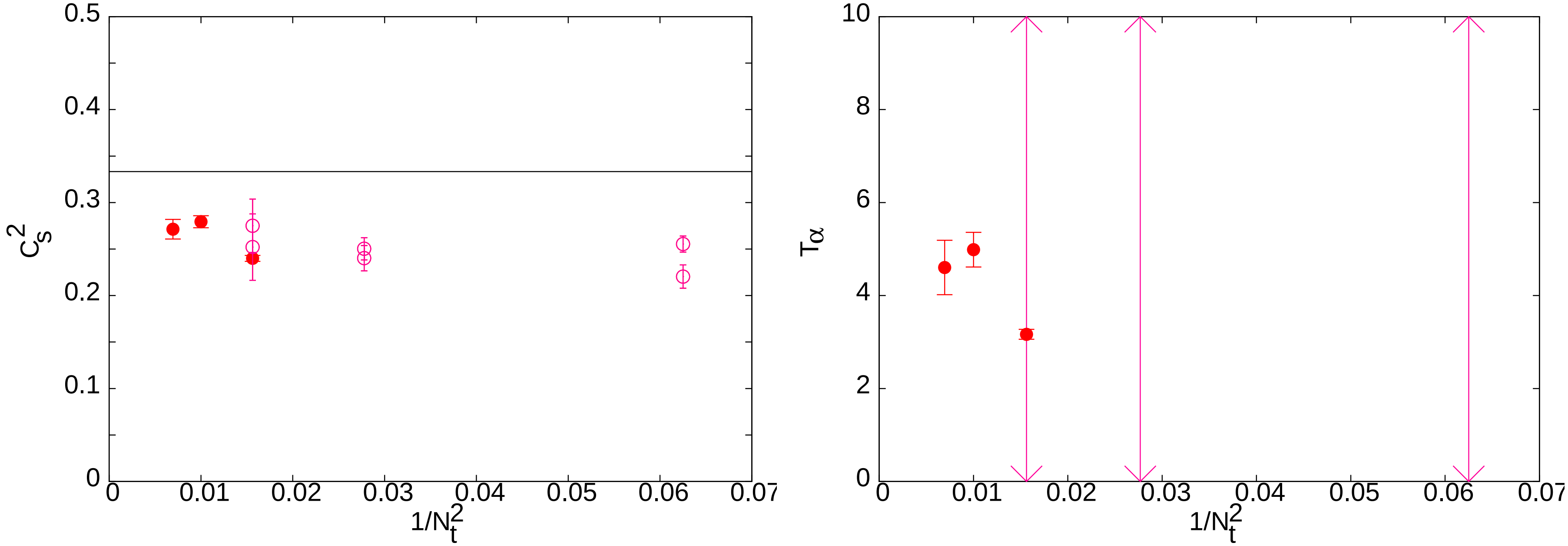
Compressibility

The inverse compressibility of an ideal gas vanishes. We have the first indication of non-ideal behaviour in the compressibility of pure gauge QCD at $T = 2T_c$.



Errors dominated by large cancellation in $\langle P_{ss} - P_{ts} \rangle$. Major improvement essentially due to increased statistics.

Speed of sound and coefficient of expansion



Approach continuum limit by taking smaller lattice spacings than all previous work. Due to necessity of also taking large statistics to keep errors under control, this is tedious work.

Summary

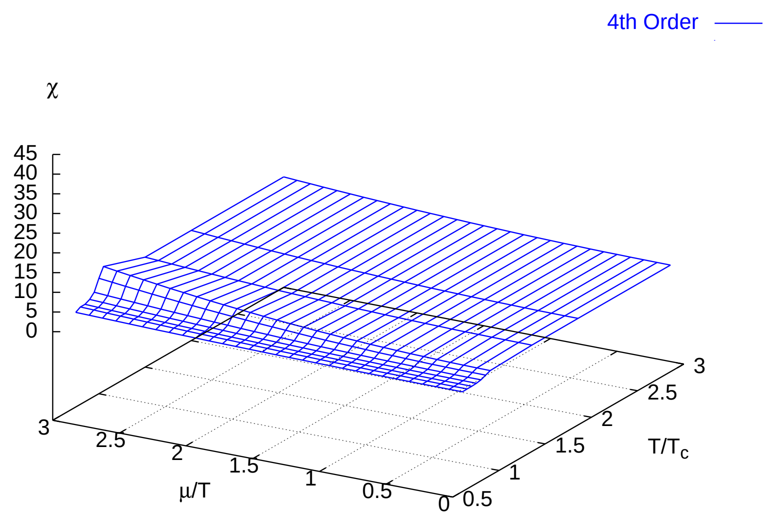
1. Analysis methods must try to identify or isolate important physical behaviour. MEM has done well in the large energy region. After subtracting the effect of the large energy regime, linear Bayesian methods show an additional bump in the small energy region.

2. A parametrised Bayesian method gives the electrical conductivity. Small time scale for transport seen. Parametrised forms check specific models of interactions and supports physics of transport.

3. Thermodynamic fluctuations can be measured with great accuracy. First indication of non-vanishing inverse compressibilty and coefficient of expansion, commensurate with departure from ideal gas values of specific heats and speed of sound.

Critical fluctuations in the QCD phase diagram

At a critical point fluctuations are unbounded. We find the critical point in the $T - \mu$ phase diagram of QCD by a Taylor series expansion of the free energy in μ with 2 flavours of dynamical quarks of mass $m/T_c = 0.1$ and with lattice spacing $a = 1/4T$.



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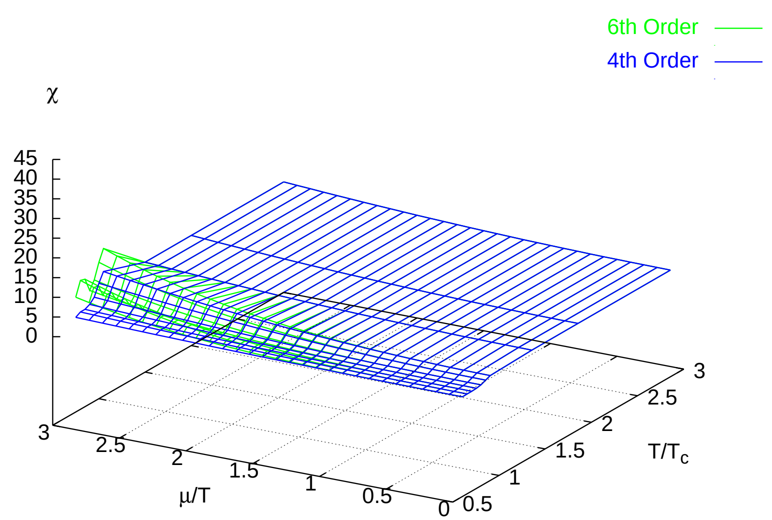

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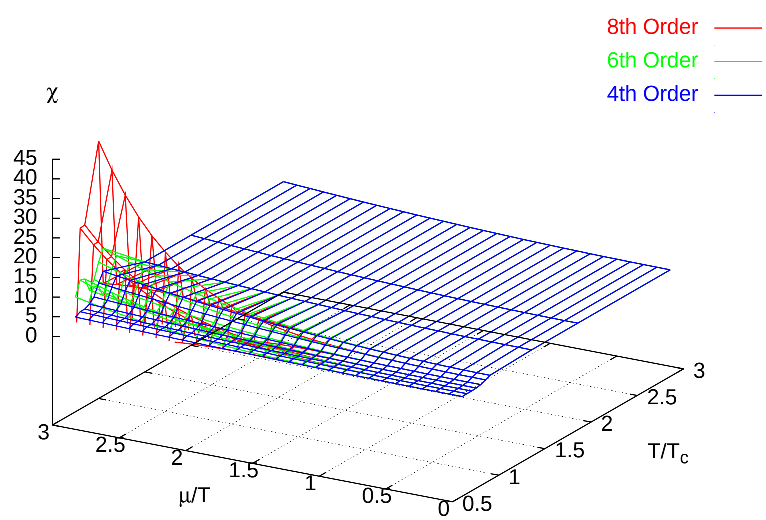

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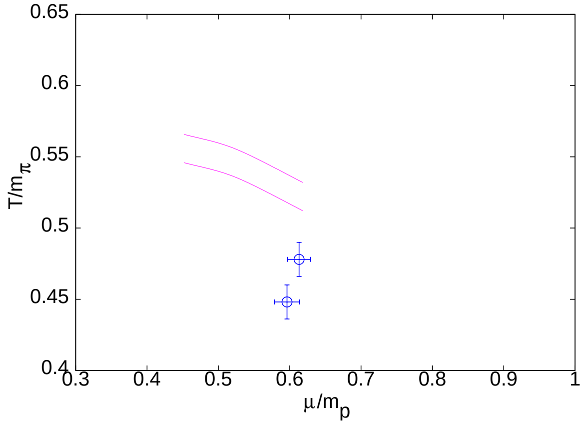

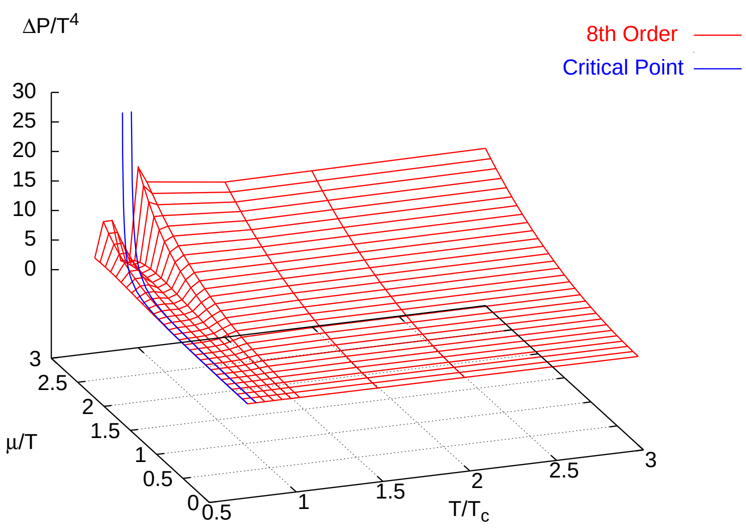

Strangeness production: Wroblewski parameter

A fluctuation-dissipation theorem connects the imaginary part of the complex susceptibility to the strangeness production rate. A Kramers-Kronig relation connects the real and imaginary parts. Then assuming that the susceptibility is given by a relaxation time formula, and that this inverse time scale is well separated from the quark mass scales, one finds that the Wroblewski parameter is the ratio of the quark number susceptibilities.