Fluctuations in Heavy-ion Collisions: signals and backgrounds

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Hot and Dense Hadronic Matter
in Relativistic Heavy-ion Collisions

University of Jammu

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The plan of these lectures

1. Fluctuations and the phase diagram: In this lecture I discuss what kind of thermal information is carried by fluctuations. I also discuss the present state of knowledge about the phase diagram of QCD.
2. Normal thermal fluctuations: I talk about the fluctuations of conserved quantities away from a phase transition, and the lattice results for these measurements.
3. Slicing up phase space: I present a critique of the various methods that experimentalists have evolved of looking at fluctuations— cutting out the messy bits and getting at the comparison with QCD.
4. Transport and balance: This lecture is about non-equilibrium processes in the plasma, such as charge (flavour) diffusion— leading to an electrical conductivity, and the influence of such processes on the balance function. This could independently check speculations about a quark-gluon liquid.

Lecture 1: Fluctuations and the phase diagram

1. Thermodynamic preliminaries— intensive and extensive variables, phase diagrams, Gibbs' phase rule.

2. The varieties of fluctuations— thermal, non-thermal, critical, etc..

3. The intensive parameters for QCD, and the structure of the QCD phase diagram, as determined by the Gibbs' phase rule.

Thermodynamics: variables

Extensive and intensive thermodynamic variables—

$$dU = TdS - PdV + \mu dN + HdM + m_\pi^2 d\phi^2 + \cdots$$

Extensive variables define the thermodynamic state of a system completely. No exceptions. Order parameters are special extensive variables. Their values distinguish between phases. They jump at a first order transition. Number of extensive variables needed is dimension of Gibbs' space— one for every conserved quantity.

Intensive variables often define the thermodynamic state of a system. Exceptions are first order transitions (i.e., phase mixtures). These variables (the complete set is needed) are the coordinates of the phase diagram— i.e., a diagram that shows the boundaries between different phases of a system.

Thermodynamics: ensembles

Canonical ensemble allows fluctuations of extensive quantities—

$$dU = TdS - PdV + \mu_B dB + \mu_Q dQ + HdM,$$

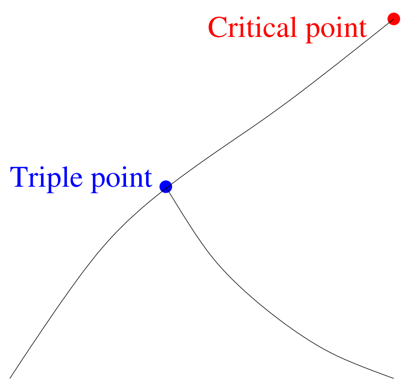
but intensive quantities are held fixed (thermostats). Phase diagram is a notion from canonical ensembles.
Microcanonical ensembles hold some of the extensive quantities fixed—

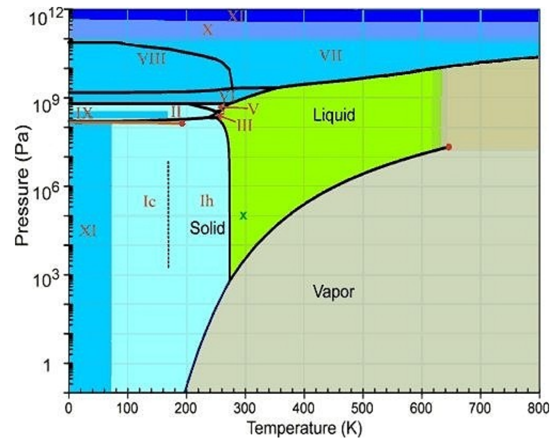
$$\begin{aligned} dU &= TdS - PdV + \mu_B dB + \mu_Q dQ, \qquad (M \text{ fixed, rotational invariance}), \\ &= TdS - PdV + \mu_B dB, \qquad (Q \text{ fixed, isospin invariance}), \\ &= TdS - PdV, \qquad (B \text{ fixed, } U_B(1) \text{ invariance}), \\ dU &= 0 \qquad (T,\, V \text{ fixed}). \end{aligned}$$

Ensemble can be microcanonical in some variables and canonical in others. "Grand canonical" can be removed from the dictionary.

Thermodynamics: Gibbs' phase rule

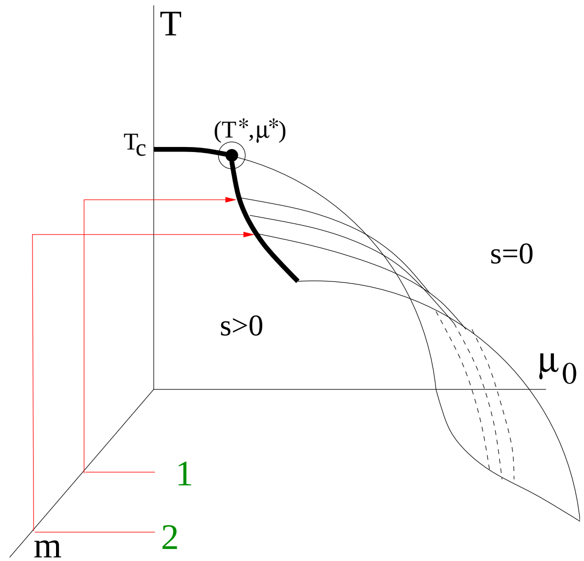
Theorem. [Gibbs] In a phase diagram of a system with G dimensions of Gibbs' space, P phases can coexist along hypersurfaces of dimension $D = G + 1 - P$.





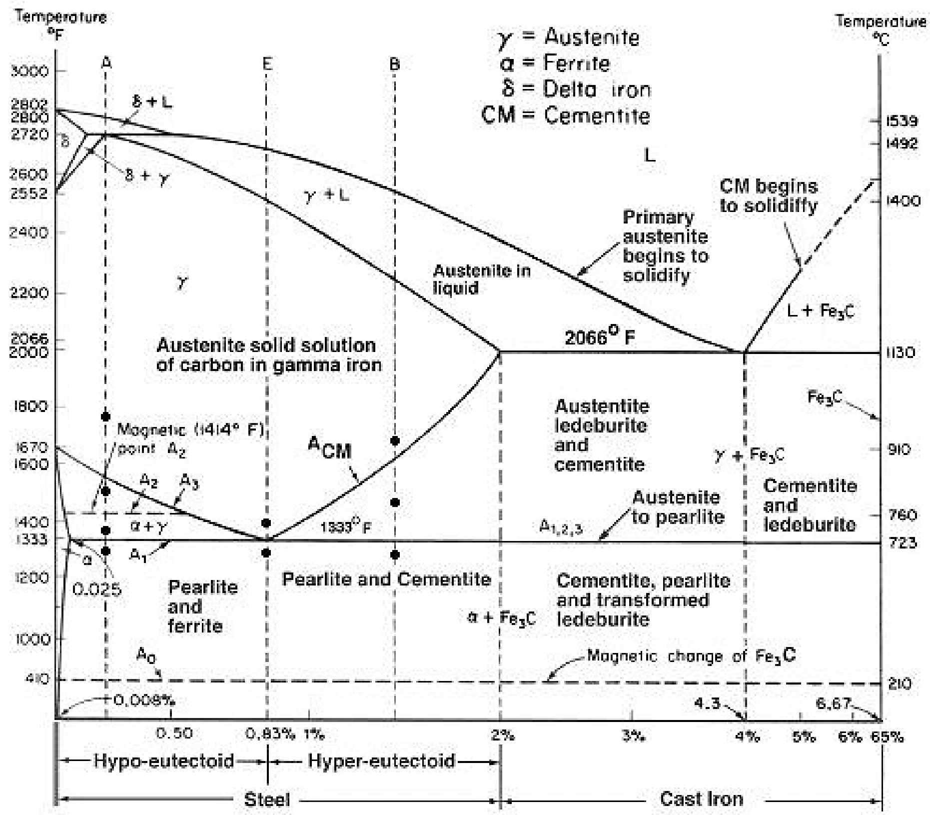
Example. [Liquids] For a one component fluid, i.e., in a two dimensional phase diagram, labelled by T and P, there are lines of first order phase transitions where two phases coexist. If there is more than one such line, then there could also be isolated triple points, where three phases can coexist. There are also critical points, which are end points of first order lines.

Thermodynamics: Gibbs' rule in QCD



Example. [Stephanov, Rajagopal, Shuryak, PRL, 81 (1998) 4816] In a three dimensional phase diagram, labelled by T, μ_B and $m = (m_u + m_d)/2$, there are surfaces of first order phase transitions, where two phases coexist. There are lines of triple points, where three phases coexist. There are also critical lines which are edges of the first order surfaces, and a tri-critical point which is the end of the triple phase line.

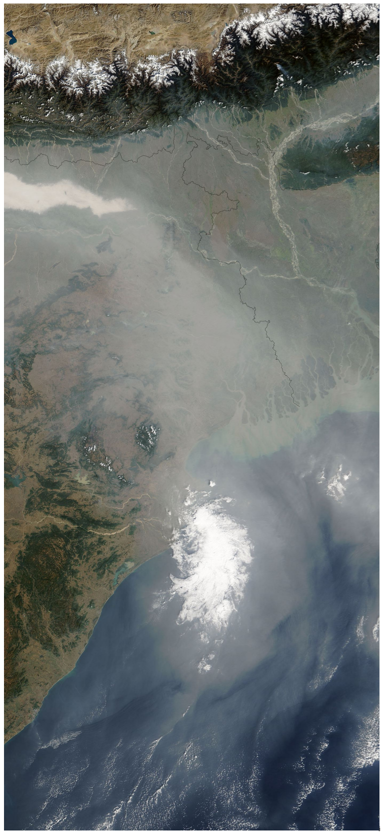
Thermodynamics: Fe/Fe$_3$C phase diagram



5% of the global economy

Not a phase diagram— x-axis is actually an intensive variable written as a density. The true phase diagram would have 3 coordinates: T, P and μ.

Summary: Thermodynamics



1. Introduced extensive and intensive thermodynamic variables.

2. Microcanonical system labelled by extensive variables.

3. Canonical system labelled by intensive variables.

4. Phase diagram shows when phase changes occur as you change the values of intensive variables.

Fluctuations and the phase diagram

- Non-thermal fluctuations due to uncontrollable changes in hadronization, impact parameter, etc.. Not of interest in heavy-ion physics now. Should remove these. Methods will include phase space cuts of various kinds.

- Thermal fluctuations are of central interest. Are they visible? Are the effects too small to be seen? If so, can the signals be boosted by some means?
  - Normal thermal fluctuations— most of the time
  - Fluctuations at a first-order phase transition— seen in low energy nuclear collisions (**Lopez et al,nucl-th/0504027**)
  - Critical fluctuations— single most important physics

Thermal fluctuations

Non-conserved quantities in matter can fluctuate randomly, but conserved quantities obey a continuity equation— $\nabla \cdot J - \partial_t \rho = 0$. So, increase in the total charge in one place implies decrease somewhere else.

Microcanonical ensemble in a variable conserves the total value of that variable. Only fluctuations are global fluctuations of the extensive variable. For 4π measurements fluctuations possible for either event-to-event value of extensive variable or slope of spectrum (T, μ etc., intensive).

Canonical ensemble in a variable is obtained by thermostatting the intensive parameter (T for energy, μ_Q for electrical charge, μ_B for baryon number, etc.). Realized in finite acceptance measurements— accepted portion is body, rest is heat-bath. T, μ etc., are fixed, and only fluctuations of extensive quantity are possible in these measurements.

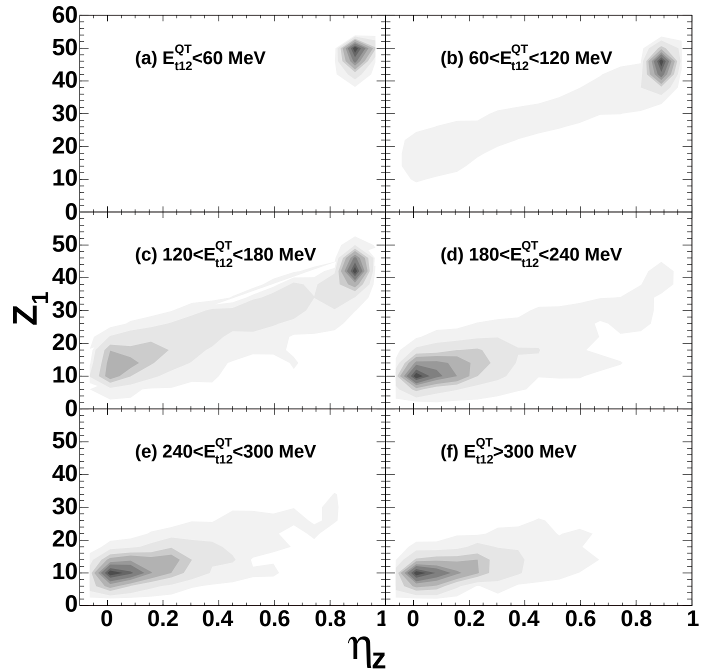
Fluctuations at a first order transition

At a first order transition two phases exist in equilibrium: phase mixture exists at T_c; called the "mixed phase". These phases have equal values of the intensive variables (T_c, P, etc.), but different values of the extensive variables (U, V, etc.). Fluctuations of extensive quantities will be seen— spinodal decomposition.

System	Quantity	Discontinuity	Observable
Water-steam	energy density	latent heat	local energy fluctuations
	mass density	specific volume	fluctuation in number
Ferromagnet	spin	spontaneous magnetization	fluctuations of local magnetization
QCD system	baryon number		baryon fluctuations
	charge		charge fluctuations
Nuclei	density		fragment size distribution

Gavin,nucl-th/9908070

Nuclear Physics Example



Pichon et al (INDRA), NP A 749 (2005) 93

Fragment charge distribution (in low-energy nuclear collisions) is bimodal, and switches from one value to another as the control parameter is changed. At the transition point the bimodal distribution gives rise to enhanced fluctuations.

http://www.g-eng.cam.ac.uk/mmc/teaching/typed/addenda/microstructures1.html

Fluctuations at a critical point

If the effective potential of the order parameter, ϕ, is of the form $V = m^2\phi^2$ (here m is called a screening mass), then the probability of a fluctuation is the Gaussian—

$$P(\phi) \propto \mathrm{e}^{-V/T} = \mathrm{e}^{-m^2\phi^2/T}$$

At a critical point, the screening mass vanishes, and hence the fluctuations are undamped. First proposed at the chiral phase transition in (unrealistic) QCD with $m_\pi = 0$. At this critical point $m_\sigma = 0$ and the fluctuations are undamped—disoriented chiral condensates.
Rajagopal and Wilczek, 1993

Idea revived when it was realized that for realistic pion masses, QCD has a critical end point at finite chemical potential.
Stephanov, Rajagopal and Shuryak, 1998

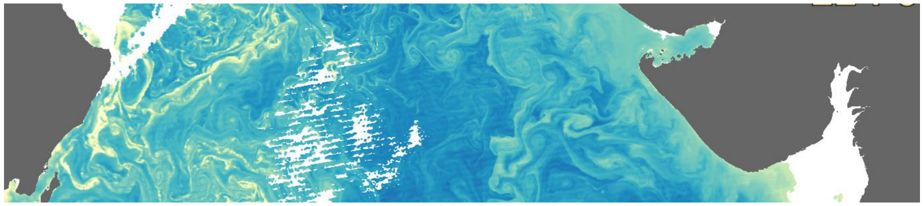
Summary: fluctuations

1. Non thermal fluctuations: not the topic of these lectures; important for understanding some aspects of hadronic physics.

2. First order fluctuations: spinodal decomposition, new avatar of dcc-like physics; important but largely unexplored. Will become important with energy scans.

3. Critical fluctuations: theoretically well understood, but application to heavy-ion collisions not yet worked out. Will become important with energy scans.

4. Normal thermal fluctuations: focus of most work these days.



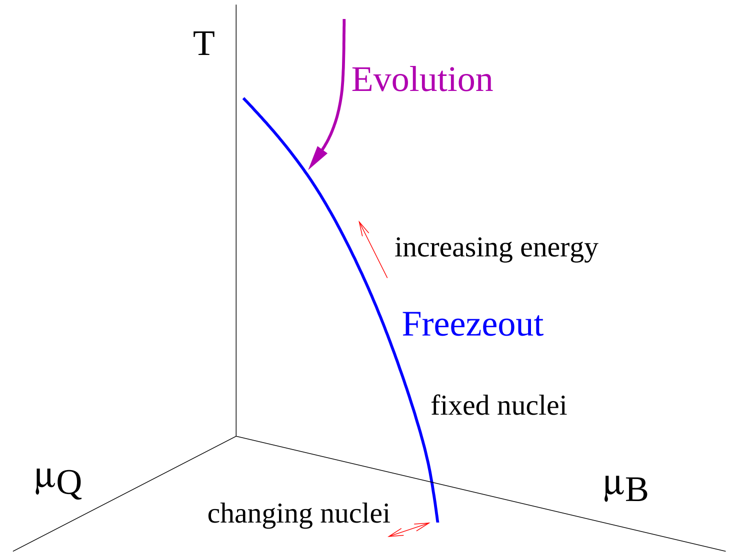
The QCD phase diagram

For $N_f = 2$, there are 5 variables—

1. the quark masses m_u and m_d ($m_\pi^2 \propto \overline{m} = (m_u + m_d)/2$).
2. the temperature T
3. the baryon chemical potential μ_B
4. the charge chemical potential μ_Q

For $N_f = 3$, there are 7 variables; all of the above and

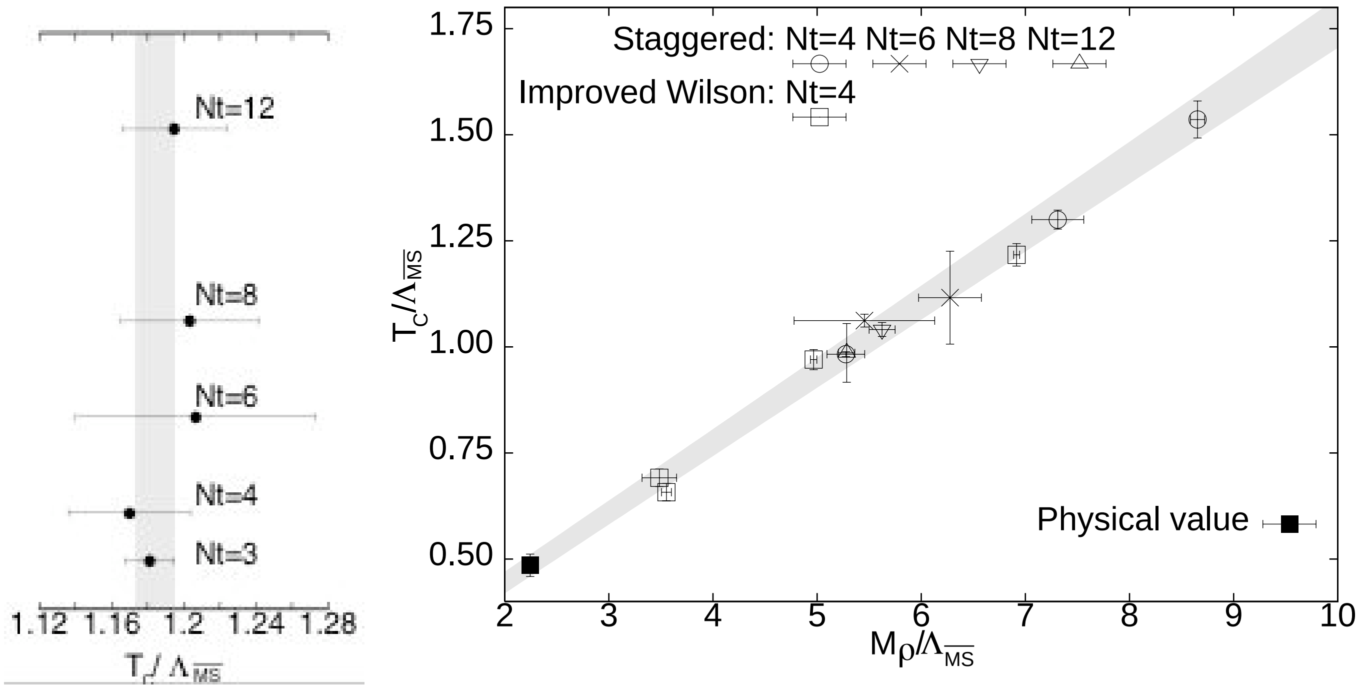
1. the quark mass m_s
2. the hypercharge chemical potential μ_Y



A curve in the T, μ_B, and μ_Q space is followed by changing collision energy— the freezeout curve. **Cleymans, Becattini, Redlich, Braun-Munzinger, etc.**

Open question: how good is the thermalization? (v_2 does not reach hydro limit?)

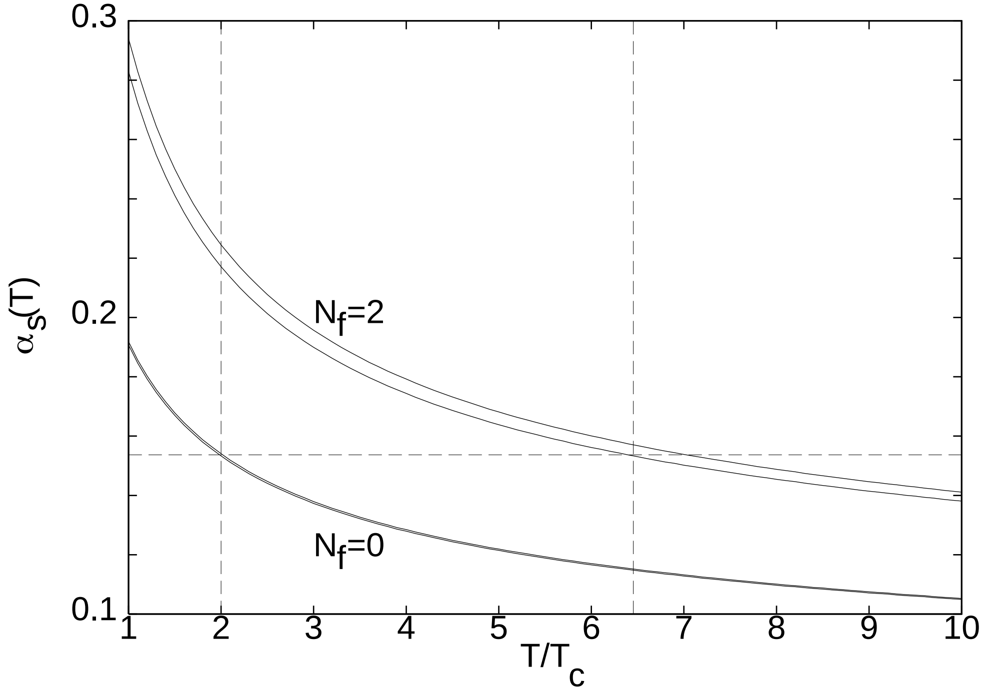
Lattice summary at the end of 2000



$T_c/\Lambda_{\overline{MS}} = 1.15 \pm 0.05$ $(N_f = 0)$ and 0.49 ± 0.02 $(N_f = 2)$

SG, PR D 64, 034507: 2001

Precision physics at finite T



$$\alpha_s(T) = \alpha_s\left(\frac{T}{\Lambda_{\overline{MS}}}\right) = \alpha_s\left(\frac{T}{T_c}\cdot\frac{T_c}{\Lambda_{\overline{MS}}}\right)$$

$\alpha_s \simeq 0.08$ means $g \simeq 1$

The T axis

Lattice work since 1985 has concentrated on exploring the T axis. Twenty years of hard work has yielded the following insight—

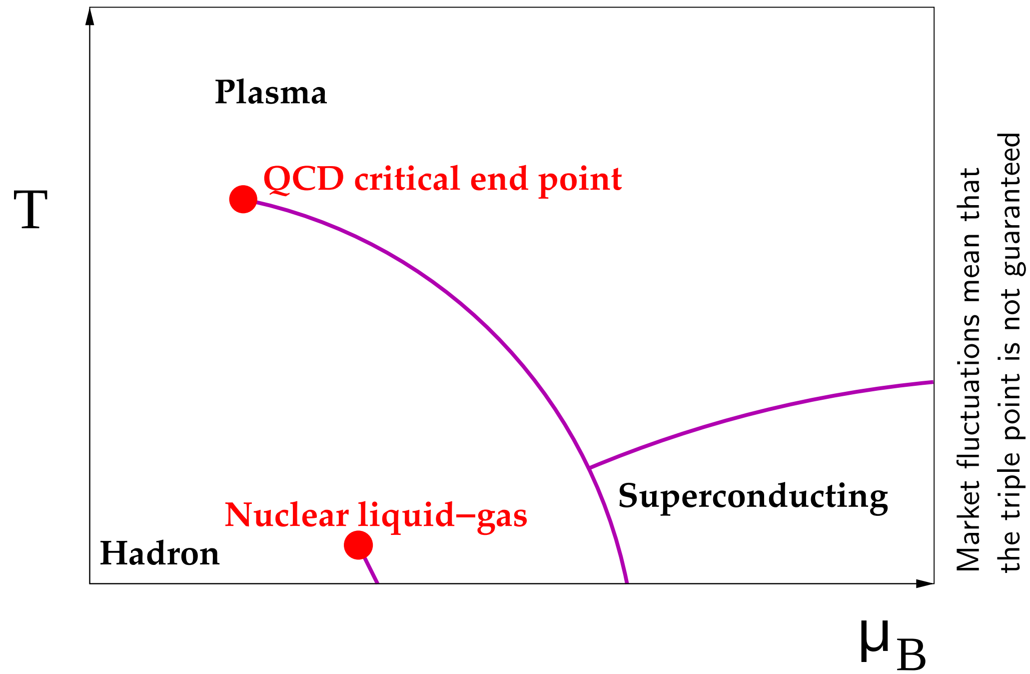
1. Quarkless QCD has a first order phase transition at $T_c = (1.15 \pm 0.05)\Lambda_{\overline{\rm MS}}$.

2. $N_f = 2$ QCD is likely to have a second order phase transition in the chiral limit ($m_\pi = 0$).

3. $N_f = 2$ QCD with realistic quark masses probably has a cross over and no phase transition in the vicinity of $T_c = 0.49\Lambda_{\overline{\rm MS}} \approx 170$ MeV.

4. $N_f = 2 + 1$ QCD with realistic quark masses probably has a cross over at T_c not very different from the above value.

The generic phase diagram of QCD: T and μ_B



Rajagopal and Wilczek,
Halasz, Jackson, et al
Others

Lattice computations of the critical end point of QCD

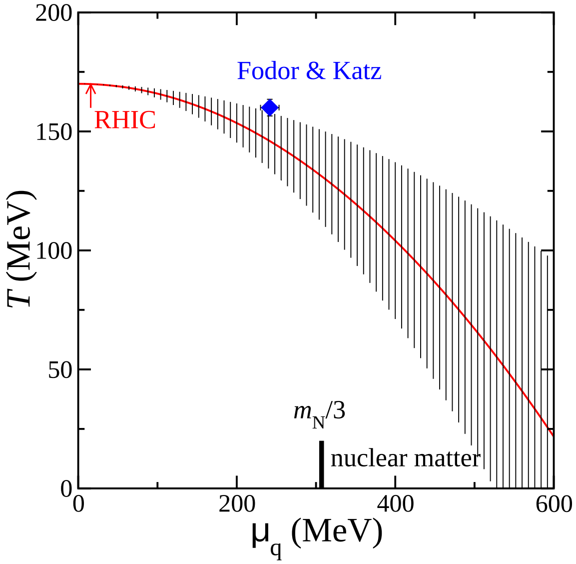
All lattice exploration of the phase diagram (T and μ_B) have several hidden parameters—

1. the lattice spacing a which can be traded for m_ρ or m_N. Taking the limit $a \to 0$ is called renormalization, or taking the continuum limit.

2. the quark mass m which can be traded for m_π. The limit $m \to 0$ is called the continuum limit. Small, non-zero m gives PCAC, approximate chiral symmetry. This universe has $m_\pi/m_\rho = 0.18$. $m_\pi/m_\rho \approx 1$ breaks chiral symmetry totally.

3. the lattice volume, $V = (aN_s)^3$, which should be infinite for thermodynamics. An internal scale for measuring lengths is the Compton wavelength of the pion. So $m_\pi a N_s$ should be large enough.

A minor gloss

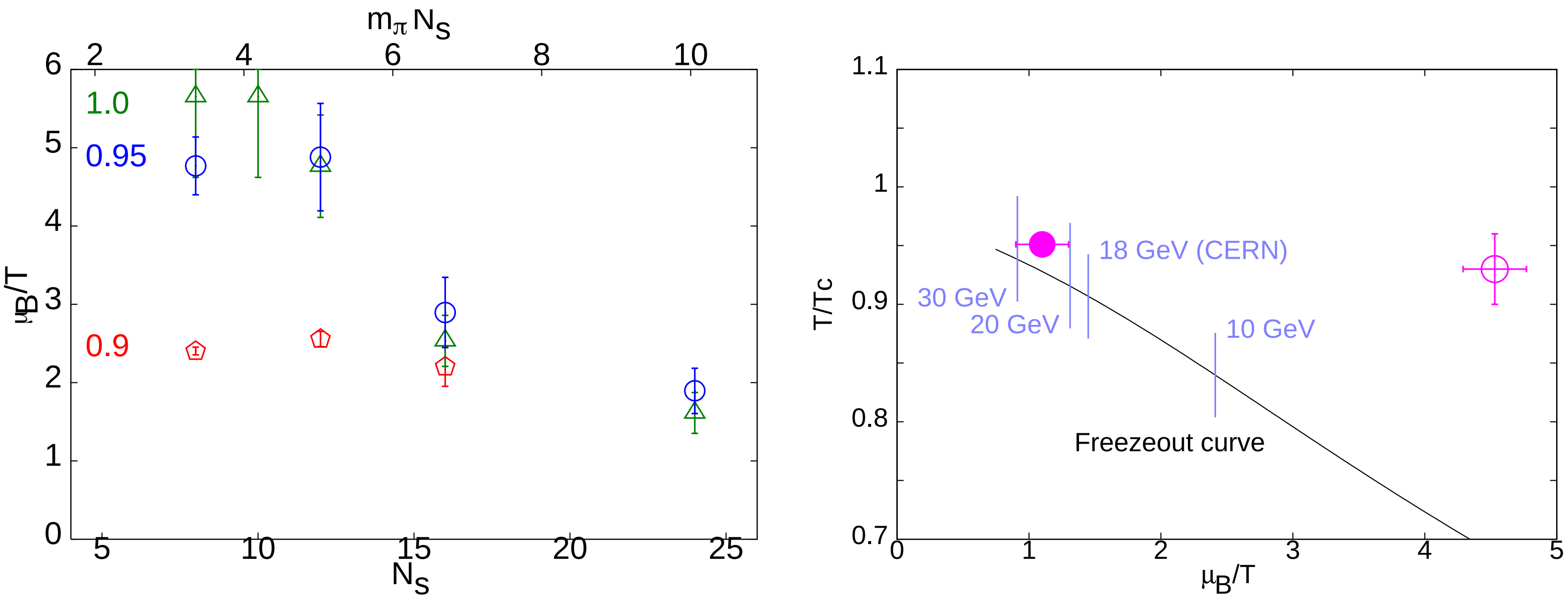


hep-lat/0402010

$\longleftarrow$ "Agreement" between Budapest and Bielefeld is the sign of a problem!

Collab	T/m_ρ	m_π/m_ρ	$m_\pi a N_s$
Budapest	5.4	0.185, 0.31	2–3, 3–4
Bielefeld	5.5	0.7	15.4
Mumbai	5.4	0.3	3.3–10.0

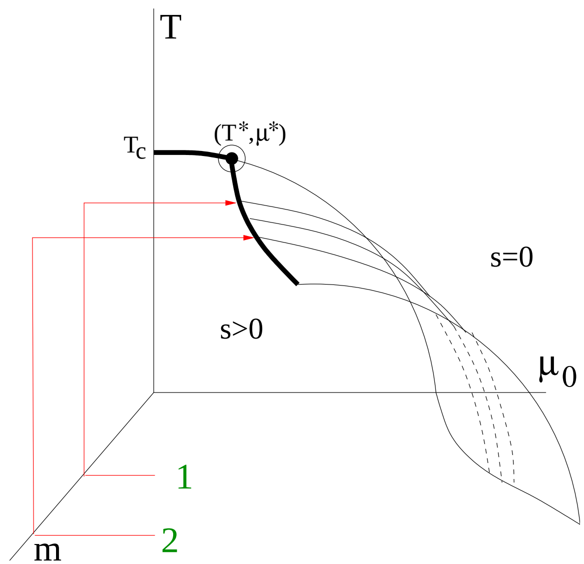
The critical end point of QCD: T and μ_B



The critical end point estimate depends strongly on $m_\pi L$ and m_π/m_ρ. Estimate on right extrapolated to $m_\pi L \to \infty$, at $m_\pi/m_\rho = 0.3$. μ^E may decrease a little for more realistic m_π.

Gavai, SG,hep-lat/0412035

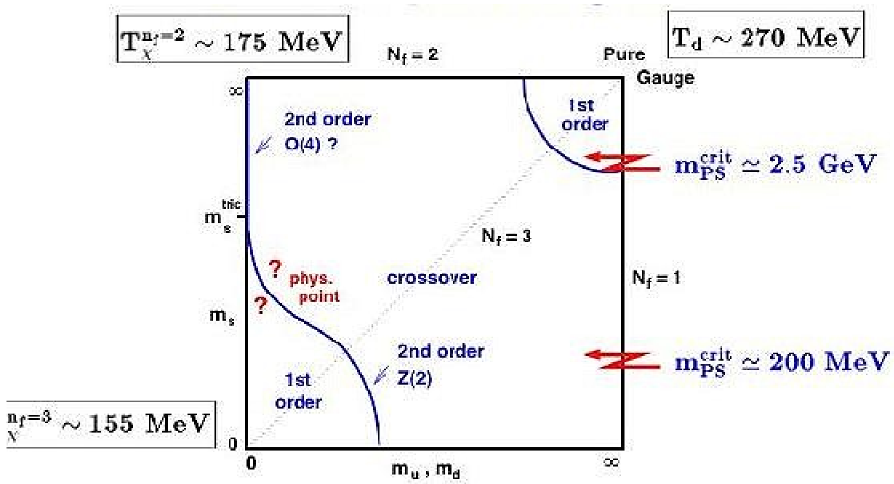
Dependence on quark mass

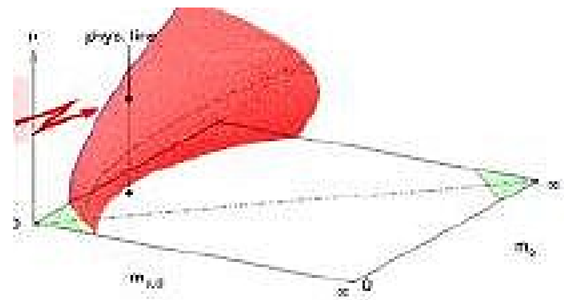


Surfaces of first order transitions with edges which are lines of critical points. These surfaces meet in lines of triple points. A tri-critical point is at the end point of this line, and therefore also at the meeting point of the two critical lines.

Note that the surface $m = 0$ has enhanced symmetry. As a result, there is a critical point if we move along the line $\mu_B = 0$ in this plane.

Adding the strange quark



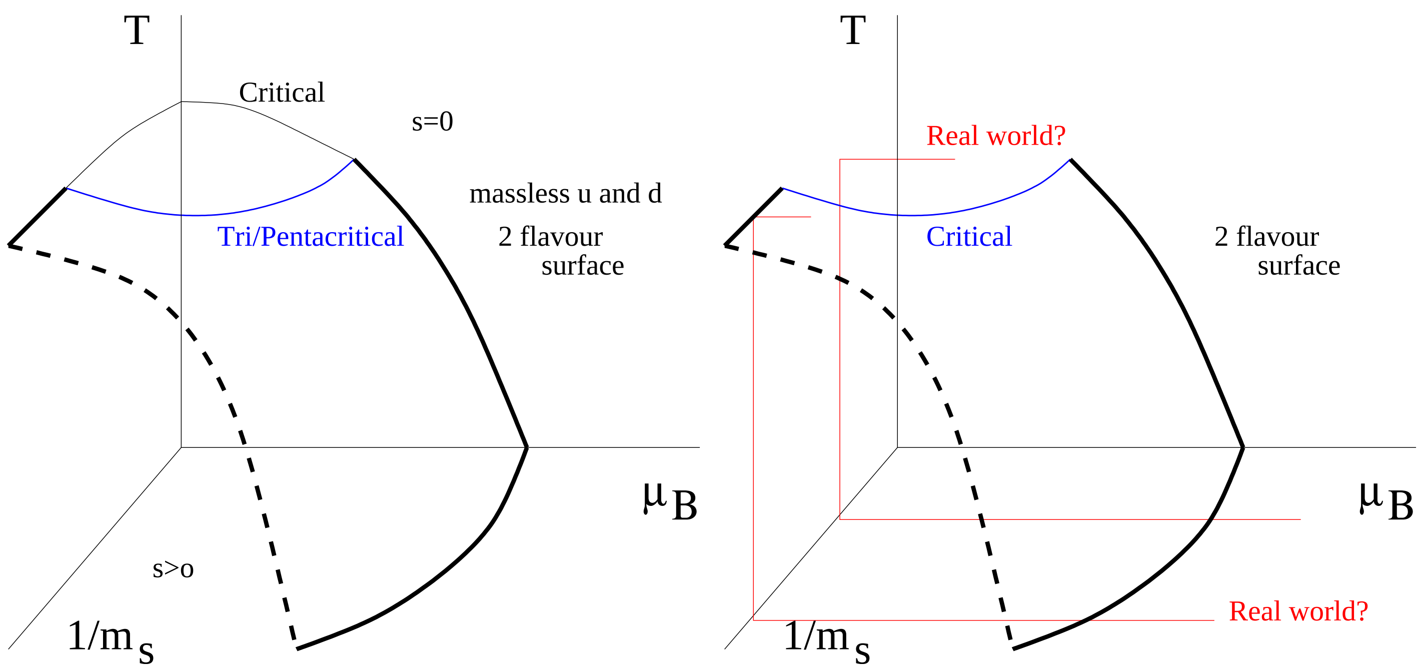


These are not phase diagrams

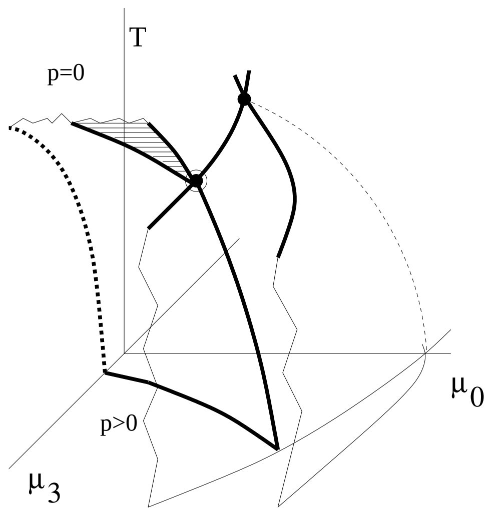
These diagrams are mnemonics— they map out the regions of quark masses where the finite temperature transition is of a given order. A phase diagram would show phase boundaries as you change parameters.

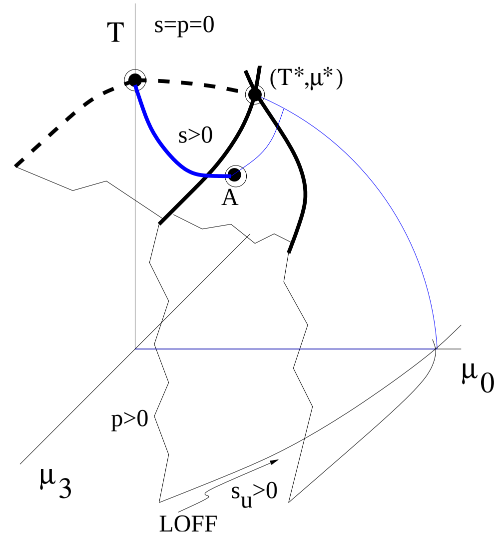
There are no phase boundaries in the $T = \mu_B = \mu_Q = \mu_Y = 0$ plane as the quark masses are varied. All over this plane chiral symmetry is broken with $\langle\overline{\psi}\psi\rangle = 0$.

Phase diagram with strange quark



Full $N_f = 2$ phase diagram





When all 5 intensive variables are taken into account, we find

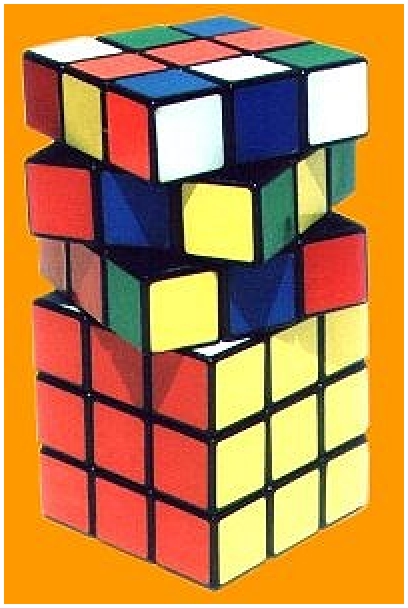
4-volumes of	2-phase coexistence
3-volumes of	normal criticality and 3-phase coexistence
surfaces of	tri-criticality and 4-phase coexistence
lines of	tetra-criticality and 5-phase coexistence
isolated	penta-criticality

Only first and second order phase transitions. What was earlier called a tri-critical point is actually a penta-critical point seen in a slice of high symmetry.

Summary: QCD phases

1. QCD phase diagram is 7 dimensional: T, 3 quark masses, 3 chemical potentials.
2. One line (T) has been explored in lattice simulations till now.
3. One plane (T and μ_B) most of the focus of experimental and theoretical work.
4. First determination of the critical point performed recently.
5. Preliminary theoretical explorations of the full phase diagram in progress.
6. Can experiments probe more than one line in the $7 - 3 = 4$ dimensional space?

Lecture 2: Normal thermal fluctuations



1. Statistical mechanics and the meaning of fluctuations; introduction to lattice computations.

2. Defining fluctuations in heavy-ion collisions, the importance of conserved quantities; a dimensionless measure.

3. Lattice measurements of susceptibility

Statistical mechanics

In the canonical distribution one computes

$$Z(T,\mu_B,\mu_Q) = \int \mathcal{D}U \mathrm{e}^{-S[U]} = \exp[-F(T,\mu_B,\mu_Q)/T].$$

All of thermodynamics is obtained in terms of derivatives. The first derivatives are the extensive variables—

$$E = T^2 \frac{\partial \log Z}{\partial T}, \quad B = T\frac{\partial \log Z}{\partial \mu_B}, \quad Q = T\frac{\partial \log Z}{\partial \mu_Q}.$$

The second derivatives are the response functions—

$$c_V = \frac{T^2}{V}\frac{\partial E}{\partial T}, \quad \chi_B = \frac{1}{V}\frac{\partial B}{\partial \mu_B}, \quad \chi_Q = \frac{1}{V}\frac{\partial Q}{\partial \mu_Q}.$$

Statistical fluctuations

$$Z(T,\mu_B,\mu_Q) = \int \mathcal{D}U \mathrm{e}^{-S[U]} = \int dx \exp\left[-\frac{(x-E)^2}{2V c_V/T^2}\right]$$

where the last integral can be written after suitable change of variables and integration over all variables except one. Furthermore, this result is obtained only when one is far from a phase transition. Similarly

$$Z(T,\mu_B,\mu_Q) = \int dx \exp\left[-\frac{(x-B)^2}{2V\chi_B T}\right] = \int dx \exp\left[-\frac{(x-Q)^2}{2V\chi_Q T}\right].$$

So normal thermal fluctuations are Gaussian and the width is a measure of the thermodynamic response functions. Note: fluctuations of extensive quantities scale as $\sqrt{V}$.

Dimensions: $E/V \propto T^4$, $B/V \propto T^3$, $Q/V \propto T^3$, $c_V \propto T^3$, $\chi_B \propto T^2$, $\chi_Q \propto T^2$.

Temperature in quantum statistical mechanics $z^{z^{z^z}}$

Since $Z = \mathrm{Tr}\,\exp[-H/T]$; and the density matrix resembles the time evolution operator, $U = \exp[iHt]$, the easy way to introduce temperature in a quantum field theory is—

- take an Euclidean-time version, $t \leftrightarrow it$
- take the size of the Euclidean-time direction to be $1/T$
- put periodic or anti-periodic boundary conditions to get the trace

Boundary conditions	values of E	poles of
periodic	$2\pi i T n$	$\frac{1}{\exp(E/T)-1}$
anti-periodic	$2\pi i T \left(n+\frac{1}{2}\right)$	$\frac{1}{\exp(E/T)+1}$

Chemical potential in quantum statistical mechanics

In quantum theory a symmetry gives a conserved charge—

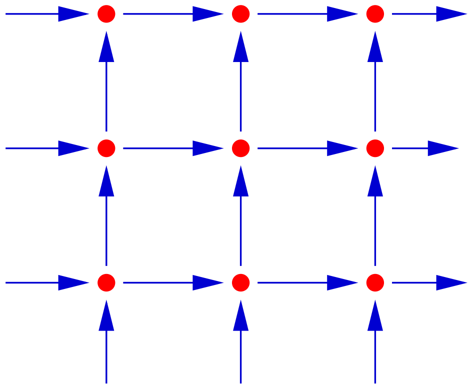
$$\text{If}\quad [H, Q] = 0 \quad \text{then for} \quad H\psi = E\psi, \quad \text{one has} \quad Q\psi = \Pi\psi.$$

Every conserved quantity adds an extensive variable to the thermodynamics. A chemical potential is added to the statistical mechanics—

$$Z(T, \mu) = \mathrm{Tr}\,\exp\left[-\frac{H}{T} - \mu Q\right].$$

A quantum field theory allows creation and destruction of particles. If particles are massive $(m \gg T)$ then particle number is conserved even without a chemical potential— energy conservation is enough. If particles are light or massless, then particle number is not conserved— only the total charge is conserved.

Lattice Statistical Mechanics



$$Z = \int dU \, d\psi d\overline{\psi} e^{-S}$$

Gauge action = discretized Maxwell equation **Wilson, Symanzik**
Quark action = discretized Dirac equation **Wilson, Kogut and Susskind, Naik, Kaplan, Neuberger and Narayanan**

$$\boldsymbol{N_t a = \frac{1}{T}}$$

Putting chemical potential on the lattice

In the Dirac equation the chemical potential enters as

$$M = \partial_\mu \gamma^\mu + m + \gamma_0 \mu.$$

This is like an imaginary scalar potential. Hence, chemical potential is put on the lattice by multiplying each Euclidean-time link matrix by $\exp \mu$.

Breaks time-reversal symmetry— distinguishes between particle and anti-particle.

As a result, $\det M$ is no longer positive, and the integral for Z cannot be done by Monte-Carlo: the fermion-sign problem.

At RHIC/LHC μ small, hence $\mu = 0$ approximation is good. Evaluating derivatives at $\mu = 0$ evades the fermion sign problem.

Summary: statistical mechanics

1. The partition function, expectation values and response functions.

2. Fluctuations: distributions of extensive variables and response functions

3. The role of conserved quantities.

4. The lattice as a computational method: temperature and chemical potential in QCD.

5. The fermion sign problem.

Fluctuations in heavy-ion collisions

Consider conserved quantities— baryon number, electric charge and hypercharge. In a microcanonical ensemble total charge is exactly conserved. But, pairs with opposite charges can be created at a point and move apart. This transport gives rise to local fluctuations. In a canonical ensemble, fluctuations in charge are due to exchange of particles with the reservoir through transport.

By the fluctuation formulae one finds that

$$\frac{\left\langle(\Delta B)^2\right\rangle}{S}=\frac{V\chi_B T}{S}=\frac{\chi_B T}{(S/V)}$$

is dimensionless. Approximate computations show large difference between two phases: in the QGP this ratio is 0.01, factor 2 larger in the HG.

Asakawa, Heinz, Muller, PRL 85, 2072: 2000

Jeon, Koch, PRL 85, 2076: 2000

Practical problem— $(\Delta Q)^2/S$ is small. Solution— comparing

$$\boldsymbol{R = \frac{N_+}{N_-}} \qquad \text{and} \qquad \boldsymbol{F = \frac{Q}{N_{ch}} = \frac{N_+ - N_-}{N_+ + N_-}}$$

one finds: $(\Delta R)^2 = 4(\Delta F)^2$.
Fluctuations of ratio dominated by fluctuations of smaller quantity (margin). Therefore—

$$(\Delta F)^2 = \left(\frac{\Delta Q}{N_{ch}}\right)^2,$$

and finally,

$$\boldsymbol{N_{ch}(\Delta F)^2 = 4\frac{(\Delta Q)^2}{N_{ch}}},$$

where the rhs is the dimensionless ratio already seen.

$A = \frac{X}{Y}$, $\frac{\Delta A}{A} = \frac{\Delta X}{X} - \frac{\Delta Y}{Y}$ or $\sqrt{\left(\frac{\Delta X}{X}\right)^2 + \left(\frac{\Delta Y}{Y}\right)^2}$

Thermal fluctuations in heavy-ion collisions

For an ideal massless quark-gluon plasma one has

$$\frac{S}{V} = (7N_cN_f + 4(N_c^2-1))T^3\frac{\pi^2}{45} \qquad \text{and} \qquad \chi_B = \frac{N_f}{9}T^2$$

This gives—

	$V\chi_B T/S$	$V\chi_Q T/S$	$V\chi_Y T/S$
$N_f = 2$	0.014	0.034	—
$N_f = 3$	0.016	0.032	0.032

More accurate determination requires the lattice.

Shuryak's suggestion— many kinds of coloured bound states in the plasma.
Koch's remark— then they show up in the ratio $\chi T/S$.

Energy fluctuations

In the same way, fluctuations of energy depend on c_V. Energy is also locally conserved, and the fluctuations are transported. The transport is ineffective if the viscosity is high, and local fluctuations can be measured.

More usually, a microcanonical definition is used. Since particle multiplicities are large, a temperature, T, can be fitted to the spectrum on an event by event basis. Then fluctuations of the spectrum are given by

$$\langle(\Delta T)^2\rangle = \langle T\rangle^2 c_V$$

Stodolsky, PRL 75, 1044: 1995

Caveat: watch out for local hot spots, jets, infrequent heavy particle production and decay. Subtract flow.

Chemistry fluctuations

The fluctuations in particle ratios such as

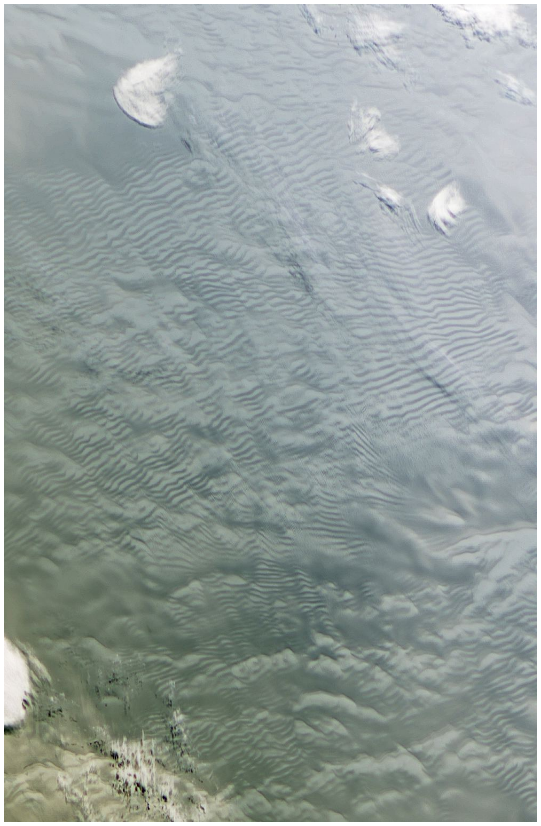
$$\frac{K^+ + K^-}{\pi^+ + \pi^-} \qquad \frac{p + \overline{p}}{\pi^+ + \pi^-} \qquad \frac{K^+}{\pi^+}$$

have been investigated experimentally.

In view of the above remarks about conserved charges, it may be more interesting to look at hypercharge fluctuations—

$$\frac{(\Delta Y)^2}{N} = \frac{\chi_Y T}{(S/V)} = \frac{\langle \Delta (p - \overline{p} + K^+ - K^- + \cdots)^2 \rangle}{\langle \pi^+ + \pi^- + K^+ + K^- + p + \overline{p} + \cdots \rangle}$$

Summary: thermal fluctuations



1. Fluctuations of charge and baryon number give information about QCD which can be compared with exact computations.

2. Fluctuations of energy are harder to do, since there are many more sources of energy fluctuations: PHENIX pursuing vigorously.

3. Fluctuations of particle ratios are interesting, but hard to compute from QCD. Fluctuations of hypercharge, Y, can be compared with computation.

Lattice predictions for fluctuations

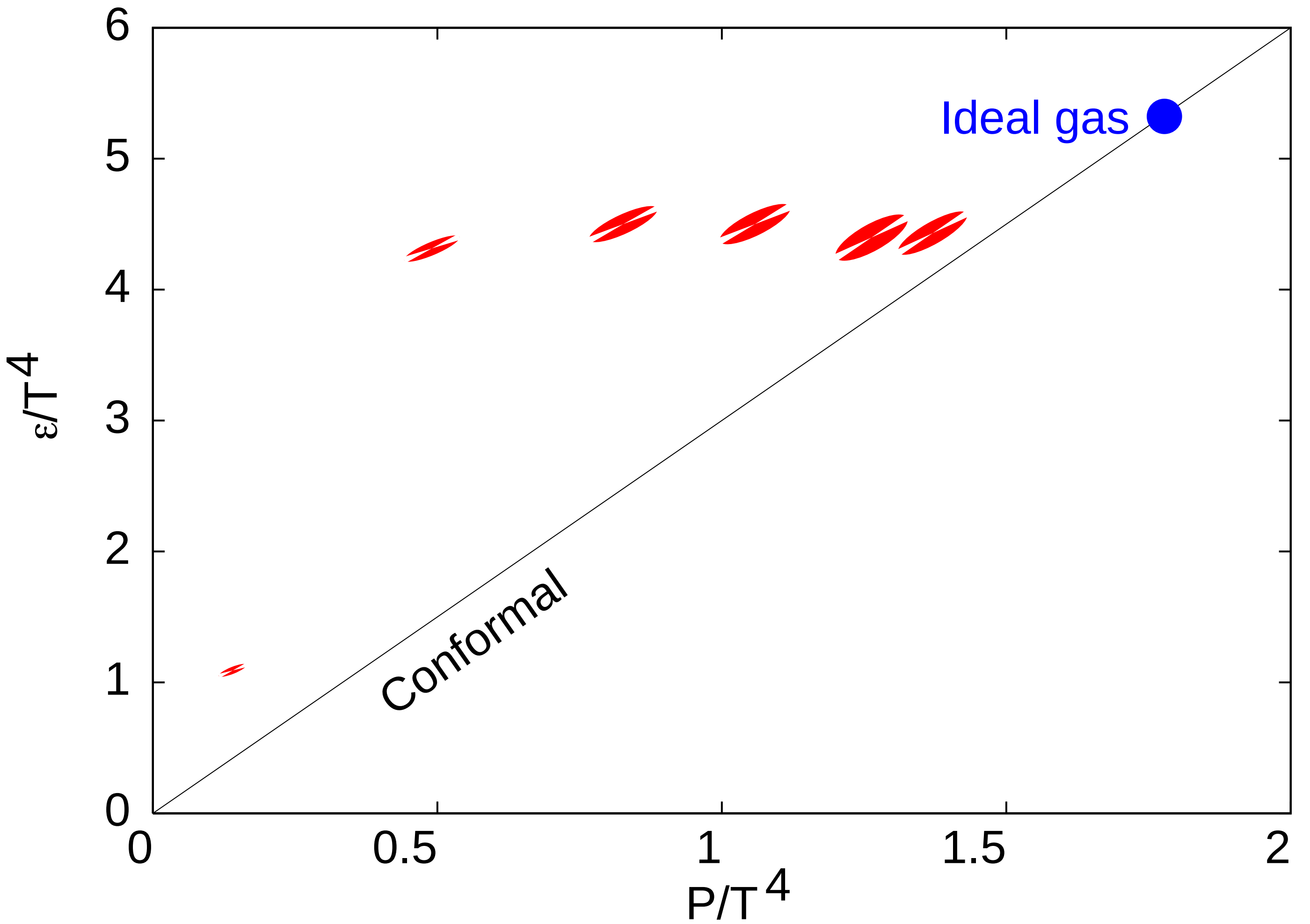
1. Equation of state of quark matter: new results.

2. Quark chemical potentials appear in the Lagrangian but the quantities that need to be computed are hadronic response functions. Simple transformations and some notation.

3. Continuum results in quenched QCD for flavour related fluctuations at vanishing chemical potential. Corrections for finite chemical potential and stability with respect to quark masses.

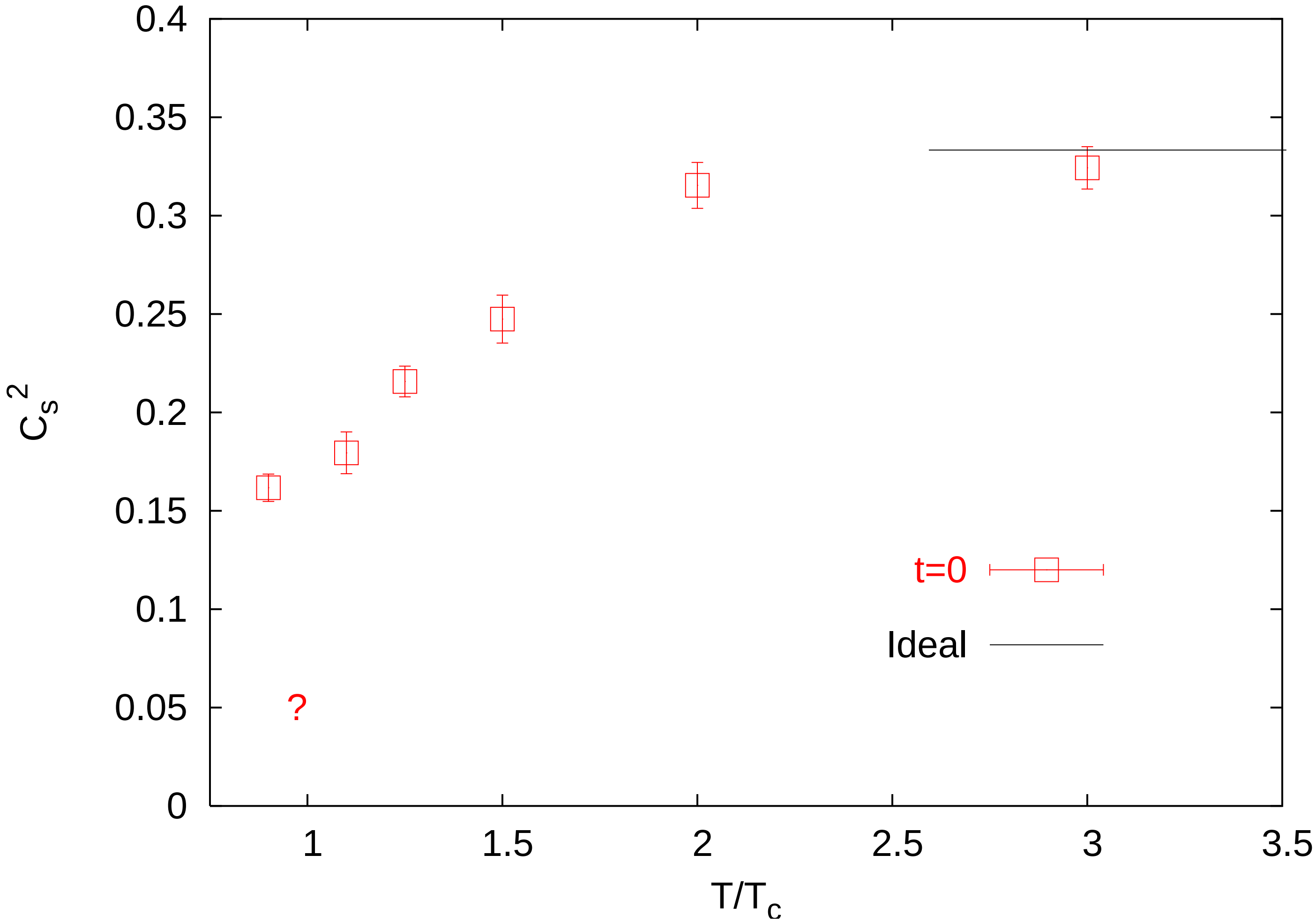
4. Energy fluctuations in the continuum limit of quenched QCD: new results.

Equation of state

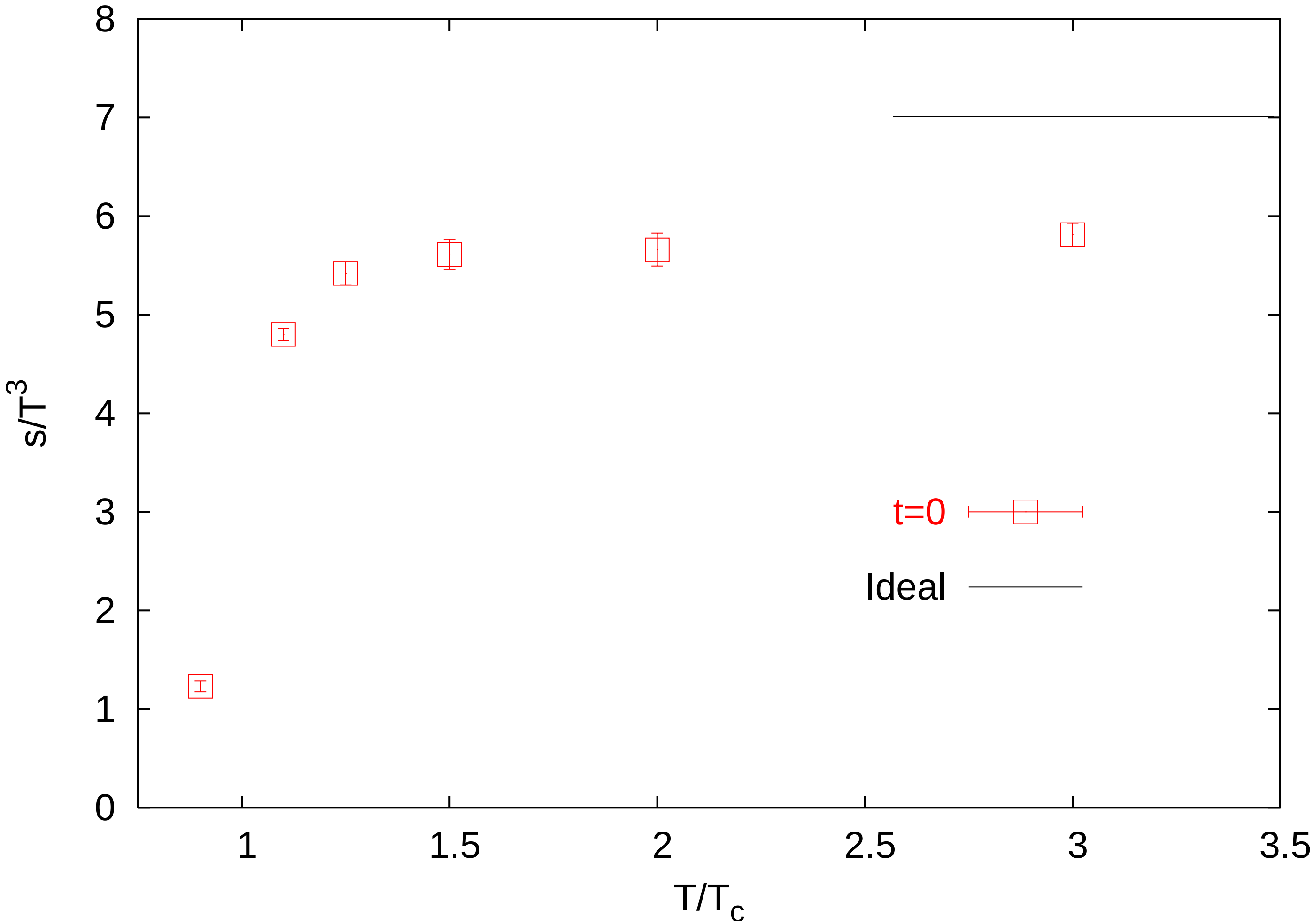


Gavai, SG, Mukherjee
PR D 71, 074013: 2005
and in progress.

The speed of sound



The entropy of the plasma



Notation: chemical potentials for quarks

Introduce a μ_f for each quark flavour f. Then the quark number density is

$$n_f = \left(\frac{T}{V}\right) \frac{\partial \log Z}{\partial \mu_f}, \qquad \chi_f = \left(\frac{T}{V}\right) \frac{\partial^2 \log Z}{\partial \mu_f^2}, \qquad \chi_{fg} = \left(\frac{T}{V}\right) \frac{\partial^2 \log Z}{\partial \mu_f \partial \mu_g}.$$

Since

$$\begin{pmatrix} B \\ Q \\ Y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} U \\ D \\ S \end{pmatrix} = M \begin{pmatrix} U \\ D \\ S \end{pmatrix}, \qquad \text{we obtain}$$

$$\chi_B = \frac{1}{9}\left(2\chi_u + \chi_s + 2\chi_{ud} + 4\chi_{us}\right), \qquad \chi_Q = \frac{1}{9}\left(5\chi_u + \chi_s - 4\chi_{ud} - 2\chi_{us}\right),$$

$$\chi_Y = \frac{1}{9}\left(2\chi_u + 4\chi_s + 2\chi_{ud} - 8\chi_{us}\right), \qquad \text{for } m_u = m_d.$$

Continuum limit for quenched QCD

	$V\chi_B T/S$	$V\chi_Q T/S$	$V\chi_Y T/S$	$V\chi_s T/S$
$1.5T_c$	0.43 (1)	0.99 (2)	0.61 (1)	0.53 (1)
$2.0T_c$	0.47 (1)	1.07 (2)	0.71 (2)	0.71 (2)
$3.0T_c$	0.49 (1)	1.09 (3)	0.71 (3)	0.84 (3)

Gavai and SG, PR D 67, 034501: 2003

Dependence on quark mass unimportant if $m \ll T$: only strange quark mass matters. The strange quark mass dependence of these results were also investigated and found to be smaller than the error bars in the table above.

SG and R. Ray, PR D 70, 114015: 2004

Less than 5% corrections when dynamical quarks are used.
Gavai, SG, Majumdar, PR D 65, 054506: 2004
MILC collaboration,hep-lat/0405029

Finite μ_B corrections

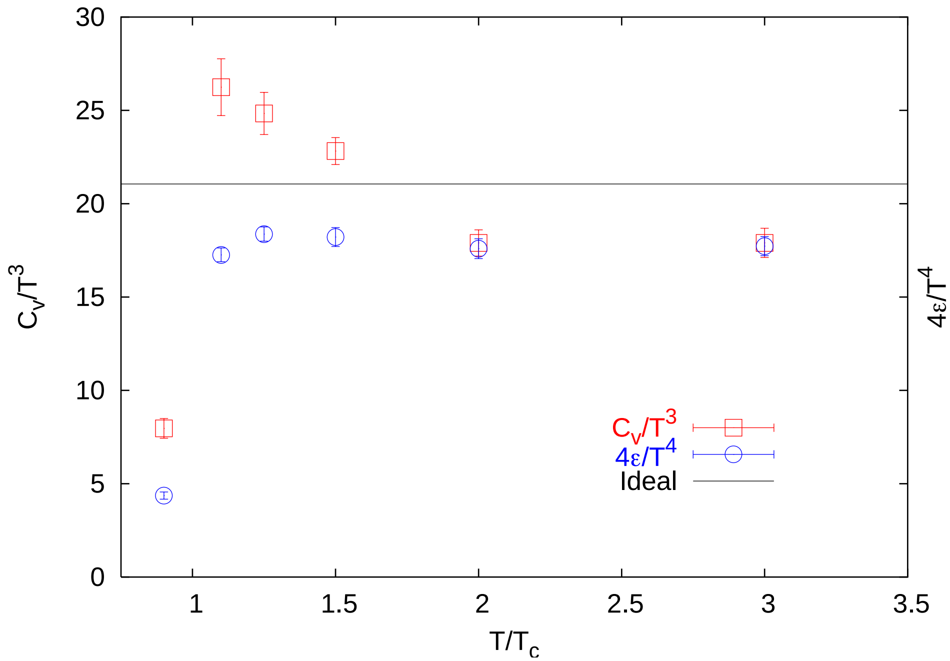
Since RHIC and SPS measurements are not made at exactly $\mu_B = 0$, one can try to estimate the corrections in a Taylor expansion of $\log Z$—

$$\chi(\mu) = \chi \left[1 + 6 \left(\frac{\mu/T}{\mu_*/T} \right)^2 + \mathcal{O}\left(\mu^4\right) \right].$$

A lattice computation found $\mu_*/T \approx 4.7$ in the range $1.5 \leq T/T_c \leq 3$. Since $\mu/T_c \leq 0.06$ at RHIC, the correction to the $\mu = 0$ result is about one part in thousand: less than the statistical error in χ.

Gavai and SG, PR D 68, 034506: 2003

Energy fluctuations



Gavai, SG, Mukherjee, PR D 71, 074013: 2005

Conformal theory has $c_V = 4\epsilon/T$, where ϵ is the energy density of the plasma.

Summary: lattice results

1. High accuracy results for $\chi_{B,Q,Y,s}$ in quenched QCD obtained at $\mu = 0$ and realistic values of the quark masses. Insensitive to changes in the quark mass, realistic changes in μ and unquenching (last change affects up to 5%).

2. High accuracy results for c_V obtained in quenched QCD. New computations with unquenching in progress.

3. All measures of fluctuations (except χ_s) rapidly approach ideal massless gas value above $2T_c$, and are in agreement with perturbation theory there. Closer to T_c, there are strong deviations from the ideal gas value. χ_s begins to approach the ideal value only after $3T_c$.

4. Since $S = (\epsilon + P)/T$ approaches ideal gas value much slower, the AHM measure $V\chi T/S$ is about 33% above the ideal gas estimate for $T \geq 2T_c$.

Lecture 3: Slicing up phase space

1. The language of multi-particle distributions and a demonstration that multi-particle correlations miss some of the physics we need.

2. Measuring c_V and a look at the methods which would see the canonical energy fluctuations: Φ, F and Σ.

3. Measuring other fluctuations and considerations of the remaining fluctuation measures: $\nu_{\rm dyn}$ and D.

Single particle inclusive distributions

If n particles of type c can be produced in a reaction with cross section σ_c^n, then one defines the inclusive cross section

$$\sigma_c^{\mathrm{incl}} = \sum_{n=1}^{\infty} n\sigma_c^n = \langle n_c \rangle \sigma_c, \qquad \text{where} \qquad \sigma_c = \sum_{n=1}^{\infty} \sigma_c^n.$$

Experimentally, one measures the momentum and energy of particles of type c, and builds the invariant distribution function—

$$f(p) = E\frac{d^3\sigma_c^{\mathrm{incl}}}{dp^3}, \qquad \text{giving} \qquad \int \frac{d^3p}{E} f(p) = \sigma_c^{\mathrm{incl}} = \langle n_c \rangle \sigma_c.$$

Bÿckling and Kajantie, Particle Kinematics, John Wiley, 1973

Multi-particle distributions

Generalize to k-particle inclusive distributions, $f(p_1, p_2, \cdots, p_k)$—

$$\int \prod_{i=1}^{k} \frac{d^3 p_i}{E_i} f(p_1, \cdots, p_k) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\sigma_c^n$$
$$= \langle n(n-1)\cdots(n-k+1)\rangle \sigma_c = \sigma_c^{k-\mathrm{incl}}.$$

Here we imagine a detector recording the energy and 3-momenta of k particles of type c. The accumulation of such observations builds up the joint distribution function $f(p_1, \cdots, p_k)$.

The factorial moments in the expressions on the right come from the fact that if we choose k particles out of n, the number of ways of doing this is the factor—

$${}^{n}P_k = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1).$$

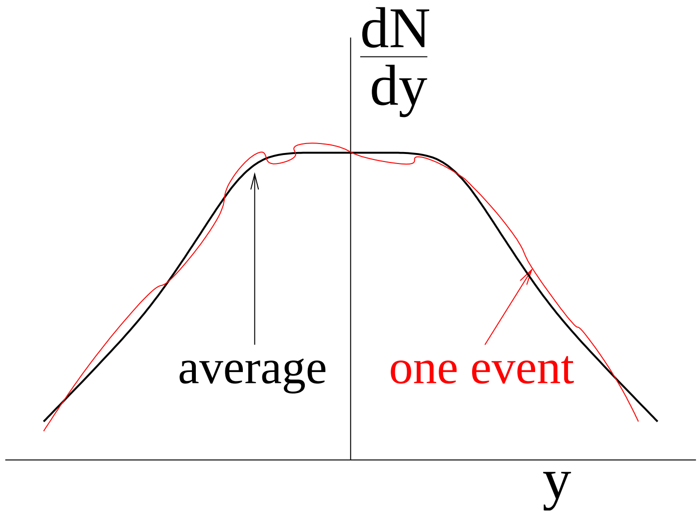
Multi-particle correlations

If the particles are produced independently, then the k-particle inclusive distribution is the product of k single-particle distributions. The k-particle correlations are defined by similarly subtracting out all the lower order correlations from the k-particle distribution—

$$\begin{aligned}
c_1(p_1) &= f(p_1) \\
c_2(p_1,p_2) &= f(p_1,p_2) - c_1(p_1)c_1(p_2) \\
c_3(p_1,p_2,p_3) &= f(p_1,p_2,p_3) - c_2(p_1,p_2)c_1(p_3) - c_2(p_1,p_3)c_1(p_2) \\
&\qquad -c_2(p_2,p_3)c_1(p_1) - c_1(p_1)c_1(p_2)c_1(p_3), \cdots
\end{aligned}$$

The properties of c_k are obtained by constructing measures which are zero if c_k vanishes. The c_k are important in trying to understand the full dynamics of QCD. They contain information on both non-thermal and thermal fluctuations.

Fluctuations and correlations



Rapidity distributions give a typical example of multiparticle correlations which could be seen as event-to-event fluctuations. The correlation function is

$$c_2(y_1, y_2) = \langle \Delta(y_1)\Delta(y_2) \rangle \,, \qquad \Delta(y) = \left.\frac{dN}{dy}\right|_y - \left\langle \left.\frac{dN}{dy}\right|_y \right\rangle ,$$

where the angular brackets denote averaging over events.

Fluctuations can remain even if correlations vanish.

Gaussian fluctuations

If $c_2 = c_3 = \cdots = c_k = 0$, then $f_k = \prod c_1$. Let $E = E_1 + E_2 + \cdots + E_k$ be the total energy; the distribution $F(E)$ is—

$$F(E) = \int \prod_{i=1}^{k} \left[\frac{d^3 p_i}{E_i} f(p_i) \right] \delta\left(E - \sum_i E_i \right)$$

$$= \int \prod_{i=1}^{k} \left[\frac{d^3 p_i}{E_i} f(p_i) \mathrm{e}^{-ixE_i} \right] \mathrm{e}^{ixE} dx = \int dx \left[\widetilde{f}(x) \right]^k \mathrm{e}^{ixE}$$

Note that $\widetilde{f}(x) = 1 + \langle E_i \rangle x + \langle (\Delta E_i)^2 \rangle x^2/2 + \cdots$, since it generates cumulants. So $f^k = 1 + \langle E \rangle x + \langle (\Delta E)^2 \rangle x^2/2k + \cdots$, which gives the central limit theorem. This tells that E is distributed as a Gaussian. Note that the fact that E is extensive was crucial to this proof.

Gaussian fluctuations can be physical and not just part of experimental errors.

Ideal gases or normal thermal fluctuations

1. At a normal point (away from all phase transitions) thermal fluctuations of extensive quantities are Gaussian. missed
2. Even an ideal gas has a specific heat, and therefore supports Gaussian fluctuations of extensive quantity. missed
3. At a second-order phase transition the correlation length of the order parameter become infinite and some non-Gaussian fluctuations could be observed.
4. At a first order phase transition, there are Gaussian fluctuations within each phase; but non-Gaussian fluctuations could be observed if the system is large enough to support multiple domains.

Health Warning

Event-by-event fluctuations are not to be treated as HBT or multi-particle correlations.

Fluctuations $\neq$ fluctuations

Measuring c_V

Specific heat could be measured through fluctuations of p_T, E_T or event-by-event fits of a temperature, T. We discuss the following measures of such canonical fluctuations—

1. The measure Φ used by NA49
2. The quantity F used by PHENIX
3. The variable Σ used by CERES
4. The measure needed but never used.

and a measure of the micro-canonical fluctuation.

The Φ measure

Let x be some property of a particle. Define $\delta x = x - \overline{x}$ where the bar is an average over the single particle inclusive distribution. Clearly, $\langle \delta z \rangle = 0$.
Using the N-particle inclusive distribution, define $\Delta x = \sum_{i=1}^{N} (x_i - \overline{x})$. Clearly $\langle \Delta x \rangle = 0$. Now define

$$\Phi(x) = \sqrt{\frac{\langle (\Delta x)^2 \rangle}{\langle N \rangle}} - \sqrt{\overline{(\delta x)^2}}.$$

Gazdzicki and Mrowczynski, Z. Phys. C 54, 127: 1992
$\Phi(E)$ **unrelated to** c_V because first term is N-particle σ normalized by N and the second term is the single particle σ. By the central limit theorem then $\Phi(E) = 0$ if there are no correlations: for example in an ideal classical gas. In a quantum ideal gas one would expect to find small values of $\Phi(E)$: positive for bosons since they all cluster together and negative for fermions since they seek different states.
Mrowczynski,nucl-th/9806089

Generalized Φ measures

Generalization proposed—

$$\Phi_n(x) = \left(\frac{\langle(\Delta x)^n\rangle}{\langle N\rangle}\right)^{1/n} - \left(\overline{(\delta x)^n}\right)^{1/n}.$$

These are combinations of the $n-th$ moments of the distribution and for $n > 2$ have no general thermodynamic or statistical significance. For symmetric distributions of x some could have special significance near a critical point.

Mrowczynski,nucl-th/9905021

Example: $\Phi_2(B)$ and $\Phi_4(B)$, where B is the baryon number, might be useful to pin down the effective potential near a QCD critical point.

The F measure

In each event let $\overline{x}$ be the average value of x and N be the number of objects over which the mean is taken. The event-to-event mean of $\overline{x}$ is $\langle\overline{x}\rangle$ and the variance is $\sigma^2(x) = \langle\overline{x}^2\rangle - \langle\overline{x}\rangle^2$. Next define

$$F = \frac{\omega_{\rm data} - \omega_{\rm baseline}}{\omega_{\rm baseline}} \quad \text{where} \quad \omega = \frac{\sigma(x)}{\langle\overline{x}\rangle}.$$

where "baseline" refers to either an experimental system or a theoretical model. PHENIX takes it as a mixed event sample.

If one defines $d = \omega_{\rm data} - \omega_{\rm baseline}$, then $\Phi = d\langle\overline{x}\rangle\sqrt{\langle N\rangle} = F\sqrt{\langle N\rangle}\sigma_{\rm baseline}$.

PHENIX, PR C 66, 024901: 2002

Since F is proportional to Φ, the two have the same problem: the relation to c_V is obscure.

The variable Σ

Define

$$\Sigma(x) = \mathrm{sgn}[\sigma_d^2(x)]\frac{\sqrt{|\sigma_d^2(x)|}}{\overline{x}}, \qquad \sigma_d^2(x) = \sigma^2(\langle x\rangle) - \frac{\overline{(\Delta x)^2}}{\langle N\rangle}$$

where angular brackets are average over a single event, σ^2 denotes variance over all events, and bar is an average over the inclusive single particle distribution.

CERES,hep-ex/0305002
Voloshin, Koch, Ritter, PR C 60, 024901: 1999

Since Σ is proportional to Φ, the two have the same problem: the relation to c_V is obscure.

The measure needed

- Since statistical fluctuations are the signal, always check whether ideal gas gives a signal or not; methods which look only for correlations are unsuited for normal thermal fluctuations.

- Canonical ensemble requires a heatbath: realize this experimentally by limited acceptance: number of particles observed, N_{obs} much less than total number of particles in the event (after removing resonance decay products).

- Canonical ensemble requires intensive variables to be fixed by thermostats; realize this by a selection on centrality to keep the energy of the system + heat-bath fixed.

- Normalization $\langle N_{\mathrm{obs}} \rangle$ needed to convert from extensive quantity $\sigma^2(E)$ to intensive quantity c_V: control over $\sigma(N_{\mathrm{obs}})$ needed.

Microcanonical fluctuations

$\Phi, \mathbf{F}, \Sigma, \nu, \mathbf{D} : \varepsilon$

With an appropriate definition of T in a microcanonical ensemble, one has—

$$c_V = \frac{\langle T\rangle^2}{\sigma^2(T)}.$$

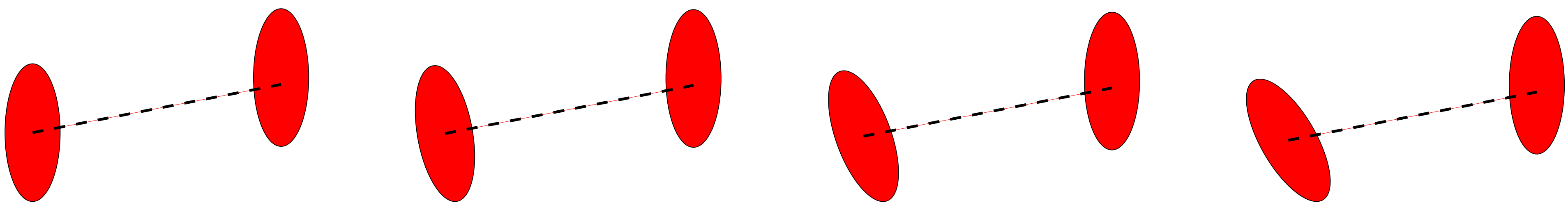
One possible definition of micro-canonical T is total energy divided by the number of degrees of freedom. **Landau and Lifschitz, Statistical Physics**

- To realize a micro-canonical ensemble full phase space should be accepted, or, number of particles not detected much less than number detected.

- In order to ensure that energy is equidistributed among all degrees of freedom, jetty events must be removed, resonance decay products must be merged or removed, effect of flow must be taken into account. Volume must be controlled.

- After this, T could be assigned to an event either by fitting a functional form, or by equipartition: see **Korus et al., PR C 64, 054908: 2001** for an analysis.

What are volume fluctuations?



The volume of the fireball is closely related to the volume of overlap of the two initial nuclei. Bins in E_T or bins in centrality are, roughly speaking, bins in the volume of the fireball. If this bin is not very small, then the ratio $\langle(\Delta V)^2\rangle/\langle V\rangle^2$ could be significant.

Volume of overlap can be traded against number of binary collisions, N_{pair}.

$$\frac{\langle(\Delta V)^2\rangle}{\langle V\rangle^2} = \frac{\langle(\Delta N_{\mathrm{part}})^2\rangle}{\langle N_{\mathrm{part}}\rangle^2}$$

Summary: measuring c_V

1. c_V is due to statistical fluctuations of total multi-particle E in the thermodynamical limit where the number of particles N is very large.

2. Methods designed to remove statistical fluctuations altogether are not useful in measuring normal thermal fluctuations, although they may be useful in finding phase transitions.

3. One needs to extract the statistical fluctuations in extensive quantities after removing the effects of volume fluctuations.

4. The events from which fluctuations are obtained must be selected very carefully to remove jets and processed with precision to delete flow effects and **reconstruct decayed resonances**.

Measuring other fluctuations

The main points about other fluctuations are the same: thermodynamical fluctuations can be uncorrelated, and therefore fluctuation measures which remove "statistical fluctuations" will miss thermodynamic information. They may still be useful for looking at phase boundaries.

Some things which arose first in the study of charge fluctuations

1. the notion of robust measures of variance

2. the (defi)notion of $\nu_{\rm dyn}$

3. the Asakawa-Heinz-Müller (AHM) measure, D (also called ω)

Robust measures of variance

Variance and covariances of quantities q_α can be expressed in terms of dimensionless ratios—

$$R_{\alpha\beta} = \frac{\langle q_\alpha q_\beta \rangle - \langle q_\alpha \rangle \langle q_\beta \rangle}{\langle q_\alpha \rangle \, \langle q_\beta \rangle}$$

The advantage of using these is that detector efficiencies may cancel out of the ratio.

Pruneau, Gavin, Voloshin,nucl-ex/0204011
Phys. Rep. 22 (Foa) and 27 (Whitmore)

Call the numerator $C_{\alpha\beta}$. The coefficient of covariance also has the same property—

$$r_{\alpha\beta} = \frac{C_{\alpha\beta}}{\sqrt{C_{\alpha\alpha} C_{\beta\beta}}}$$

$\nu_{\rm dyn}$

$$\boldsymbol{\nu_{\rm dyn} = R_{++} + R_{--} - 2R_{+-}}$$

This is again a correlation measure, and hence insensitive to thermal fluctuations. It can be shown that $\mu_{\rm dyn}$ is proportional to Φ, and hence they have the same problem.

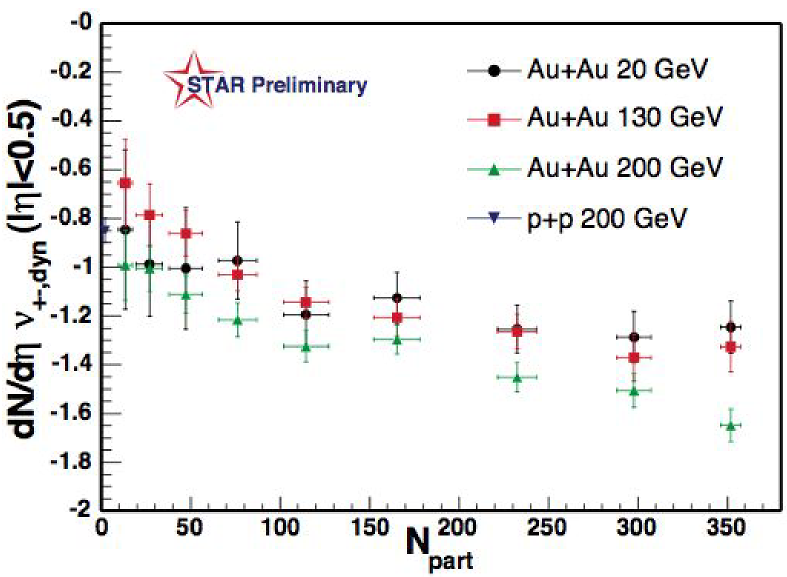
If an AA collision is equivalent to $N_{\rm part}$ independent collisions, then it can be shown that—

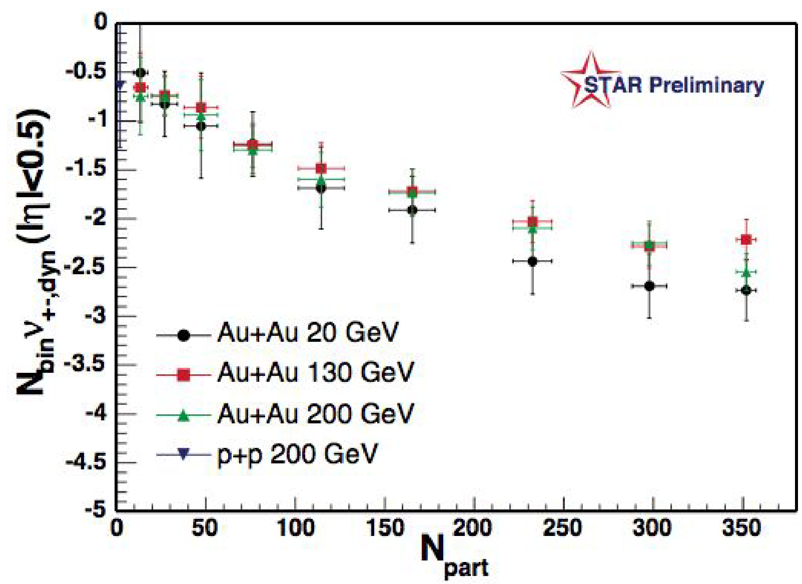
$$R^{AA}_{\alpha\beta} = \frac{R^{NN}_{\alpha\beta}}{\langle N_{\rm part}\rangle} + \frac{\langle(\Delta N_{\rm part})^2\rangle}{\langle N_{\rm part}\rangle^2}, \qquad \text{therefore} \qquad \nu^{AA}_{\rm dyn} = \frac{\nu^{NN}_{\rm dyn}}{\langle N_{\rm part}\rangle}.$$

If there are final state interactions, then the effective number of independent events decreases. Hence this scaling of $\nu^{AA}_{\rm dyn}$ can be taken as an measure of thermalization.

Fluctuations show approach to thermalization

Φ, F, Σ, ν, D : ω



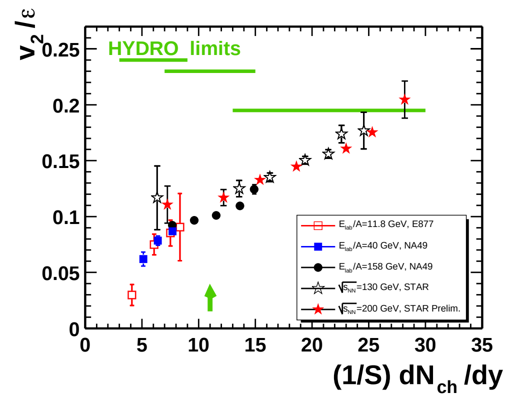


Note control over $N_{\rm part}$, i.e., volume variations.

Westfall (STAR),nucl-ex/0404004

Note concordance with v_2.

NA49,nucl-ex/0303001



Reduced variance: the AHM measure

Asakawa-Heinz-Müller propose

$$\boldsymbol{D \equiv \omega = \frac{4\langle(\Delta Q)^2\rangle}{\langle N\rangle}}$$

which has the property of being able to see thermal fluctuations. It was pointed out that

$$D = D\Big|_V + \frac{4(\langle N_+ - N_-\rangle^2}{\langle N_+ + N_-\rangle}\left(\frac{\langle(\Delta V)^2\rangle}{\langle V\rangle^2}\right) = D\Big|_V\left[1 + \frac{\langle(\Delta V)^2\rangle}{\langle V\rangle^2}\right]$$

and hence volume fluctuations may distort the signal.

Fix this using data.

Slicing up the acceptance for D

It would be good to make the following analysis—

1. **Select all events within the most central 10% bin.**
2. **In this set find D by slicing into thinner bins of width δb.**
3. **If the i-th bin gives D_i, then define the mean**

$$\overline{D(\delta b)} = \frac{\sum_i N_i^{ev} D_i}{\sum_i N_i^{ev}},$$

   i.e., by giving weight proportional to the number of events, N_i^{ev}, in that bin.
4. **Plot $\overline{D(\delta b)}$ as a function of $(\delta b)^2$ and extract the estimator $\overline{D(\delta b = 0)}$.**

If weak or no dependence is seen, it means that the volume fluctuation term $\langle(\Delta V)^2\rangle/\langle V\rangle^2$ is under control. Otherwise, the intercept at $\delta b = 0$ is the correct estimate of D. Restricting to the most central 10% also minimizes flow effects.

Summary

- Φ, F, Σ and $\nu_{\rm dyn}$ see only correlated fluctuations; therefore are not direct measures of the quark number susceptibilities. However, when constructed with the baryon number, they may be good probes of the phase boundary.

- $\nu_{\rm dyn}$ is a probe of thermalization, and gives results in concordance with those obtained using v_2. The latter indicates that hydro sets in only at the highest centralities. The former shows that the number of effective sources drops as the number of participants rises. (**Gavin**) Both are relatively independent of $\sqrt{S}$ at fixed centrality.

- Since D (i.e., ω) is sensitive to the QNS, it is worthwhile trying to control this measure further.

- An algorithm has been given to construct a quantity $\overline{\boldsymbol{D(\delta b = 0)}}$ which is an estimator of D corrected for volume fluctuations.

Lecture 4: Transport and Balance

As the fireball expands and cools, does it remember the fluctuations of early times? If it does, then it is not in equilibrium: so we need to study relaxation to equilibrium. Lattice QCD gives the mean free path of quarks. Balance functions can search for this information.

1. Non-equilibrium statistical mechanics: transport coefficients and mean free times; the liquid-like nature of the plasma and cross checks.

2. Balance functions and their uses.

Non-equilibrium phenomena: transport coefficients

Charge susceptibility is the zero-momentum limit of the EM current correlator—

$$\chi_q(T) = -\frac{1}{2\pi}\Pi_{00}(\omega = 0, \boldsymbol{q} \to 0), \qquad \Pi_{\mu\nu}(\omega, \boldsymbol{q}) = i\left\langle \boldsymbol{T} J^{EM}_\mu(q) J^{EM}_\nu(0)\right\rangle_T$$

Unpolarized photon production involves the imaginary part—

$$(2\pi)^3\omega\frac{\partial^7 N}{\partial^4 x\partial^3 q} = \frac{e^2}{\exp(\omega/T) - 1}\mathrm{Im}\,\Pi^\mu_\mu(\omega, \boldsymbol{q}),$$

at lightlike momenta $\omega = |\boldsymbol{q}|$. The zero-momentum limit gives the electrical conductivity—

$$\sigma(T) = \frac{1}{6}\;\frac{\partial}{\partial\omega}\mathrm{Im}\,\Pi^i_i(\omega, \mathbf{0})\bigg|_{\omega\to 0}$$

where the sum is over spatial polarizations only.

A finite conductivity implies that the soft photon production rate vanishes.

The electrical conductivity and mean free time

Lattice measurements show that

$$\frac{\sigma(T)}{TC_{EM}} = 7.5 \pm 0.8, \qquad C_{EM} = 4\pi\alpha \sum_f e_f^2$$

A Drude formula relates this to the mean free time of quarks—

$$\sigma = \frac{C_{EM} S_q \tau_q}{m}$$

which then gives $\tau_q = 0.3$ fm within 10% errors. The charge diffusion constant can then be found using the kinetic theory formula—

$$D = \tau_q c_s^2 \approx 0.1 \text{ fm}$$

SG, PL B 597,57: 2004

The quark-gluon liquid

From the lattice measurement, one can also construct $\tau_g \approx 2\tau_q$ and then use kinetic theory to get $\eta/S \approx 0.21$ to compare with the AdS/CFT prediction $1/4\pi$. If η and other transport coefficients are small, like in a nearly ideal liquid, then diffusion coefficient is small, mean-free times are small, and systems relax quickly to equilibrium.

A measure of liquid-like behaviour is the dimensionless ratio

$$\boldsymbol{\ell = \tau S^{1/3}}.$$

In a non-relativistic fluid this would measure the mean free path in units of the interparticle spacing. For gases the values are large, for liquids, $\ell \simeq 1$. In perturbation theory, $\tau \propto 1/Tg^4 \log(1/g)$, $S \approx 32\pi^2 T^3/45$, and hence ℓ becomes large, since g is small. In reality, for $T \approx 2T_c$, $S \approx 4T^3$, and hence $\boldsymbol{\ell = 0.81}$, implying that the plasma is liquid-like.

SG, Pramana, 61, 877: 2003, (QCD 2002: IIT-K)

Check all earth-shaking results

Small η/S implied by preliminary analysis of spectra and flow using blast-wave approximation to full hydrodynamics. Check this—

1. Use second-order dissipative hydro to compute spectra and flow to check the stability and extract a more accurate value of η. (See, for example, **Chaudhuri, Heinz,nucl-th/0504022**)

2. Look for photon dimming, and use it to compute the value of σ. This should have concordance with η. **SG,hep-ph/0411355**

3. Try to extract quark diffusion constant from fluctuation measurements. The rapidity width of a charge fluctuation is $\Delta Y \leq 2D/\tau_c$, for fluctuations created at time τ_c. Fluctuations at early times should be seen in bins of small rapidity: $\Delta Y \leq 0.4$ for $\tau_c = 0.5$ fm. **Aziz, Gavin, PR C 70, 034905: 2004**

The balance function

Define the conditional probability that a particle of type b has momentum p_2 if also a particle of type a has momentum p_1—

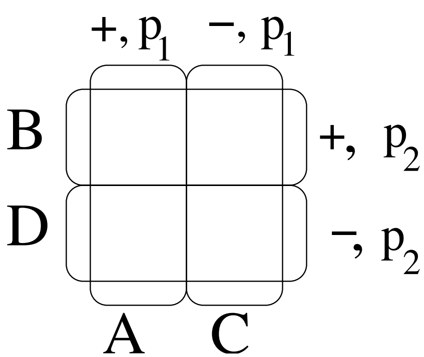
$$p(b, p_2|a, p_1) = \frac{N(a, p_1; b, p_2)}{N(a, p_1)}.$$

The balance function is defined in terms of these conditional probabilities as—

$$\boldsymbol{B(p_2|p_1) = \frac{1}{2}\left[p(b, p_2|a, p_1) + p(a, p_2|b, p_1) - p(b, p_2|b, p_1) - p(a, p_2|a, p_1)\right].}$$

If $b = -a$ and global charge is conserved, then the normalization condition is $\sum_{p_2} B(p_2|p_1) = 1$. This can be checked by noticing that $1 = \sum_{p_2}[p(-a, p_2|a, p_1) - p(a, p_2|a, p_1)$, being the probability that if a is seen at p_1, then somewhere there is an excess of $-a$ exactly sufficient to cancel that.
Bass, Danielewicz, Pratt, PRL 85,2689: 2000

Understanding the balance function



Let a and b be charges; then there are 4 types of events. Clearly, A and C are mutually exclusive and B and D are mutually exclusive. If $\mu_Q = 0$, then we can write $\boldsymbol{p(B|A) + p(D|A) = p(B|C) + p(D|C) = \mathcal{P}}$. Using charge symmetry further, we can write

$$\boldsymbol{p(B|A) = p(D|C) = p(p_2, p_1)} \qquad \text{and} \qquad \boldsymbol{p(D|A) = p(B|C) = q(p_2, p_1)}$$

where $p + q = \mathcal{P}$. The balance function can be written as

$$\boldsymbol{B(p_2|p_1) = q(p_2, p_1) - p(p_2, p_1).}$$

The "normalization condition" is the charge weighted probability, $1 = \sum_{p_2}(q - p)$. If we write $\sum_{p_2} \mathcal{P}(p_2, p_1) < 1$. to allow for neutrals, then the two normalizations together imply the obviously wrong conclusion $\sum_{p_2} p(p_2, p_1) < 0$!

Analyzing conditional probabilities

The conditional probabilities are $p(B|A) = p(B\&A)/p(A)$, so that,

$$\sum_{p_2} \mathcal{P}(p_1, p_2) = \frac{\sum_{p_2}[p(B\&A) + p(D\&A)]}{p(A)} = \frac{\sum_{p_2} p[(B \cup D)\&A]}{p(A)}$$

If $B \cup D$ were independent of A, then the last expression would become $\sum_{p_2} p(B \cup D) < 1$ (to allow for neutrals). However, global charge conservation implies a correlation between the events, which is expressed as the normalization of the balance function—

$$p(A) = \sum_{p_2}[p(B\&A) - p(D\&A)]$$

So the two conditions together imply $\sum_{p_2} \mathcal{P}(p_2, p_1) > 1$, allowing both p and q to be positive. However, at least one of them has normalization exceeding unity.

A disagreement about the balance function

A suggested use of the balance function is to ask the following question: if one sees a charge q at any point p_1 in phase space, what is the probability that the opposite charge is seen at a rapidity separation Δy, i.e., compute $B(\Delta y|p_1)$. The first estimate used an independent fragmentation model in the hadronic phase.
Bass, Danielewicz, Pratt, PRL 85,2689: 2000

The use of an absolute coordinate, p_1, for one and a difference, Δy, for the other is questioned. Clearly, care is required, and proper Jacobian factors must be included in order to do this, as they must be also to integrate over part of the phase space for p_2. But these are text-book problems.

The use of an independent fragmentation model in the hadronic phase has been questioned. While this may be an useful starting point, years of hadron phenomenology has indicated that this requires correction. In particular for a sensitive tool like the balance function a more realistic model is needed.
Trainor,nucl-ex/0301122

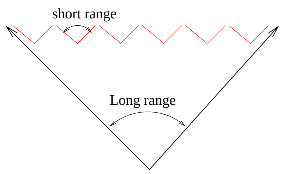
Hadronic balance function

It is more realistic to use a string model (such as Lund) which incorporates event-by-event strong charge anti-correlations with a range ΔY. A toy model with rapidity range $[-Y:Y]$ and only short ranged correlations—

$$p(\Delta y, p_1) = \begin{cases} \frac{1}{\Delta Y} \\ 0 \\ \frac{1}{Y-2\Delta Y} \end{cases} \qquad \text{and} \qquad q(\Delta y, p_1) = \begin{cases} 0 & (|\Delta y| < \Delta Y), \\ \frac{2}{\Delta Y} & (\Delta Y < |\Delta y| < 2\Delta Y), \\ \frac{1}{Y-2\Delta Y} & (\text{otherwise}). \end{cases}$$

The balance function is also short ranged but can be negative

$$B(\Delta y|p_1) = \begin{cases} -\frac{1}{\Delta Y} & (|\Delta y| < \Delta Y), \\ \frac{2}{\Delta Y} & (\Delta Y < |\Delta y| < 2\Delta Y), \\ 0 & (\text{otherwise}). \end{cases} \qquad \text{reflecting}$$



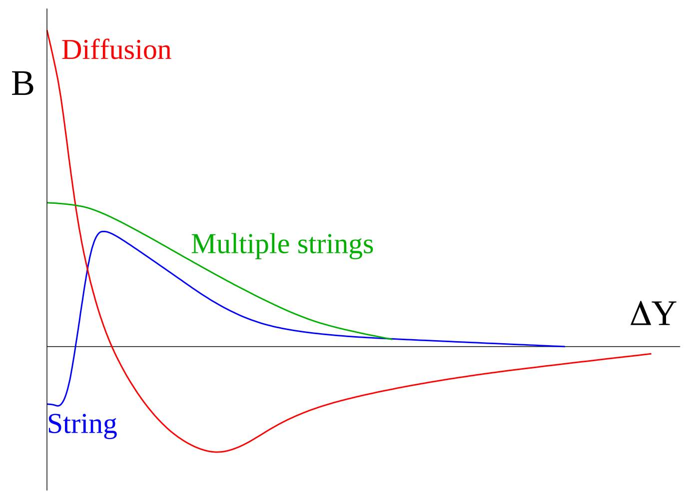
This structure will be lost in a dense medium. **SG, Parikh, PL B 219, 354: 1989**

Signals of diffusion

Diffusion gives rise to charge anti-correlations. A toy model of diffusion can also be built up in the same way as before—

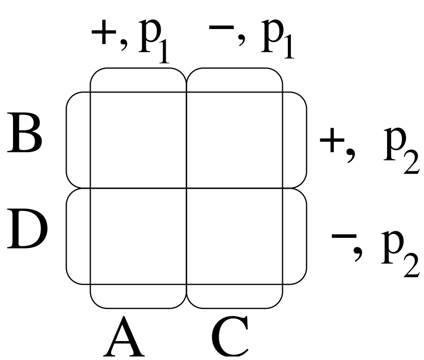
$$p(\Delta y, p_1) = 3\sqrt{\frac{\sigma'}{2\pi}}\mathrm{e}^{-y^2/2\sigma'}, \qquad \text{and} \qquad q(\Delta y, p_1) = 4\,\sqrt{\frac{\sigma}{2\pi}}\mathrm{e}^{-y^2/2\sigma},$$

where σ' is related to χ_Q and σ to D, and $\sigma \ll \sigma'$ since the diffusion coefficient is very small.



Two caveats

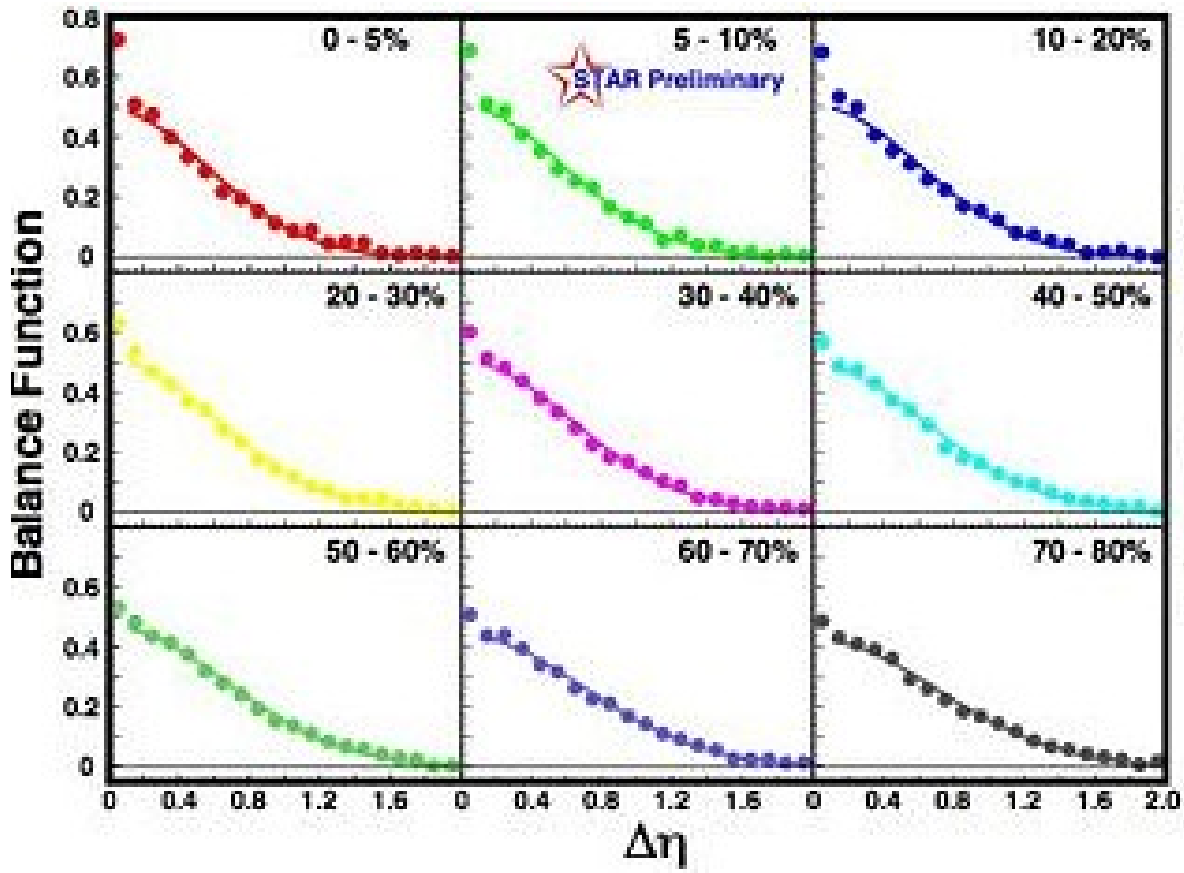
Limited acceptance changes the normalization, $\sum_{p_2} B < 1$, since a balancing charge could lie outside the range of acceptance. Also, for $B(\Delta y|p_1)$, the distribution in Δy may not be uniformly covered if p_1 lies near the edge of the acceptance range.



At finite chemical potential the system can no longer be specified with just two conditional probabilities. One has $p(A) = 1/(exp[(E_1 - \mu)/T]\pm)$ and similarly for the other marginal probabilities, such as, $p(B) = \sum_A p(B|A)p(A)$.

When such considerations place a strong limitation on the use of the balance function, it would be more useful to construct the correlation function directly.

Star preliminary



Why the bump at small Δy for the most central events? Not string effect, because it doesn't appear in peripheral events. Is it diffusion or an acceptance effect?

Summary: balance function

1. Balance function defined to look at charge balance inside the acceptance range. Useful only for $\mu = 0$, when the average charge is zero, and for large acceptance.

2. Balance function can be negative: single string fragmentation is an example. Observed positive balance function then shows that dense matter is formed (**SG, Parikh, PL B 219, 354: 1989**).

3. Could be used to measure diffusion constant of charges. Is there any evidence for a short diffusion constant in the balance function? (**STAR preliminary**).

4. If indeed diffusion constant is small, then fluctuations can only be measured from the time of chemical freeze out.

Course summary

- Basic theory is well known: QCD. Experiments are now very precise.
- Why is there still no quantitative match between theory and experiment?
- **Because experiments and theory talk about different things.**
- Important to make sure that experiments measure what theory gives.

Create one baseline test of
non-perturbative QCD in heavy-ion collisions:
FLUCTUATIONS