A transport coefficient: electrical conductivity

Sourendu Gupta (TIFR, Mumbai)

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1. Why the electrical conductivity?
2. Extraction of spectral function from lattice data using Bayesian methods.
3. Using lattice data to find the time scale of transport phenomena.

Why electrical conductivity

- Transport coefficients are extracted from correlation functions of conserved quantities. All transport coefficients give interesting physics. So choose a correlator which is easy to measure on the lattice.

- We choose the electromagnetic vector correlator. This is known to be one of the easiest to measure.

- It is related to many pieces of interesting phenomenology such as— skin depth of soft photons, diffusion coefficients for charge, baryon number and strangeness.

- With some assumptions, this measurement can be used to estimate the viscosity.

Linear Response Theory

The response, $\mathbf{A}(t)$, of a system to a force $\mathbf{F}(t)$ if non-linear terms are neglected—

$$\mathbf{A}(t) = \int_{-\infty}^{\infty} dt' \chi(t-t')\mathbf{F}(t') \qquad \text{hence} \qquad \mathbf{A}(\omega) = \chi(\omega)\mathbf{F}(\omega).$$

Causality implies $\chi(t) = 0$ for $t < 0$. As a result $\chi(\omega)$ is regular in the upper half plane and dispersion relations follow. The spectral density is the imaginary part of $\chi(\omega)$ as ω approaches the real axis from above. A microscopic computation explicitly relates $\chi(\omega)$ to the retarded propagator. From this follow the Kubo formulæ relating the transport coefficient and the zero energy limit of the spectral density—

$$\chi \propto \lim_{\epsilon\to 0} \int d^3x' \int_{-\infty}^{t} dt'' \mathrm{e}^{\epsilon(t''-t)} \int_{-\infty}^{t''} dt' \langle \mathbf{A}(\mathbf{x},t)\mathbf{A}(\mathbf{x}',t')\rangle.$$

J. Hilgevoord, *Dispersion Relations and Causal Description*, North-Holland, 1960

Temporal correlators: electrical conductivity and photon emissivity

The differential photon emissivity is given by—

$$\omega \frac{d\Omega}{d^3p} = \frac{C_{EM}}{8\pi^3} n_B(\omega;T) \rho^\mu_\mu(\omega, \mathbf{p}; T) \qquad \text{where} \qquad C_{EM} = 4\pi\alpha \sum_f e_f^2 \approx \frac{1}{21}.$$

In terms of the DC electrical conductivity ($\mathbf{j} = \sigma \mathbf{E}$)

$$\sigma(T) = \frac{C_{EM}}{6} \left. \frac{\partial}{\partial\omega} \rho^i_i(\omega, \mathbf{0}; T) \right|_{\omega=0}, \qquad \frac{8\pi^3\omega}{C_{EM}T^2} \frac{d\Omega}{d^3p} = 6\,\frac{\sigma}{T}.$$

Since $k^\mu \rho_{\mu\nu} = 0$, we have $\rho_{00} = 0$ along the line $\mathbf{p} = 0$. Formally,

$$\rho_{00}(\omega, \mathbf{0}; T) = 2\pi\chi_Q \omega\delta(\omega),$$

where χ_Q is the charge susceptibility.

Lattice Correlators

In the (Euclidean) lattice theory one constructs equilibrium correlation functions which are related to the spectral function by—

$$G(\tau, \mathbf{p}; T) = \int_0^\infty \frac{\omega}{2\pi} K(\omega, \tau; T) \rho(\omega, \mathbf{p}; T).$$

In a lattice theory there are N_t points in the τ direction, but there is a continuous infinity of ω.

Replace integral by sum, the linear relation above becomes a set of linear equations: more variables than equations. Inverse of K is ill defined. Convert to a minimisation/Bayesian problem.

Another case where the inverse matrix is ill-defined is when there are more equations than unknowns. In this case the usual method of solution is by least squares.

Regularisation

When the number of variables is larger than the number of equations, maximize the Bayesian probability—

$$P(\rho|G) \propto P(G|\rho)P(\rho) = \exp[-F(\rho)],$$

$$F(\rho) = (G - K\rho)^T \Sigma^{-1} (G - K\rho) + \beta U(\rho)$$

β is a regularisation parameter, Σ is the covariance matrix of the measured G, and $U(\rho)$ is a function which we are free to choose. This function encodes our prior knowledge of the system.

A. N. Tikhonov and V. Y. Arsenin, *Solutions of Ill-posed Problems*, Wiley, New York (1977)

1. Specifying a regulator function is not, in principle, different from parametrising $\rho(\omega)$. In practise, the results may differ.

2. When the errors are large then Σ^{-1} is small and the prior assumptions effect the solution strongly. When the errors are small then Σ^{-1} is large and improper assumptions can sometimes be identified and consequently removed.

Flavours of regularisation

1. Maximum Entropy Method has $U = \sum \rho \log(\rho/\rho_0) - \rho$, where ρ_0 is a free further choice.
   Y. Nakahara, M. Asakawa and T. Hatsuda, *Phys. Rev.*, D 60 (1999) 091503, QCD TARO, *Nucl. Phys.* B (Proc. Suppl.) 63 (1998) 460

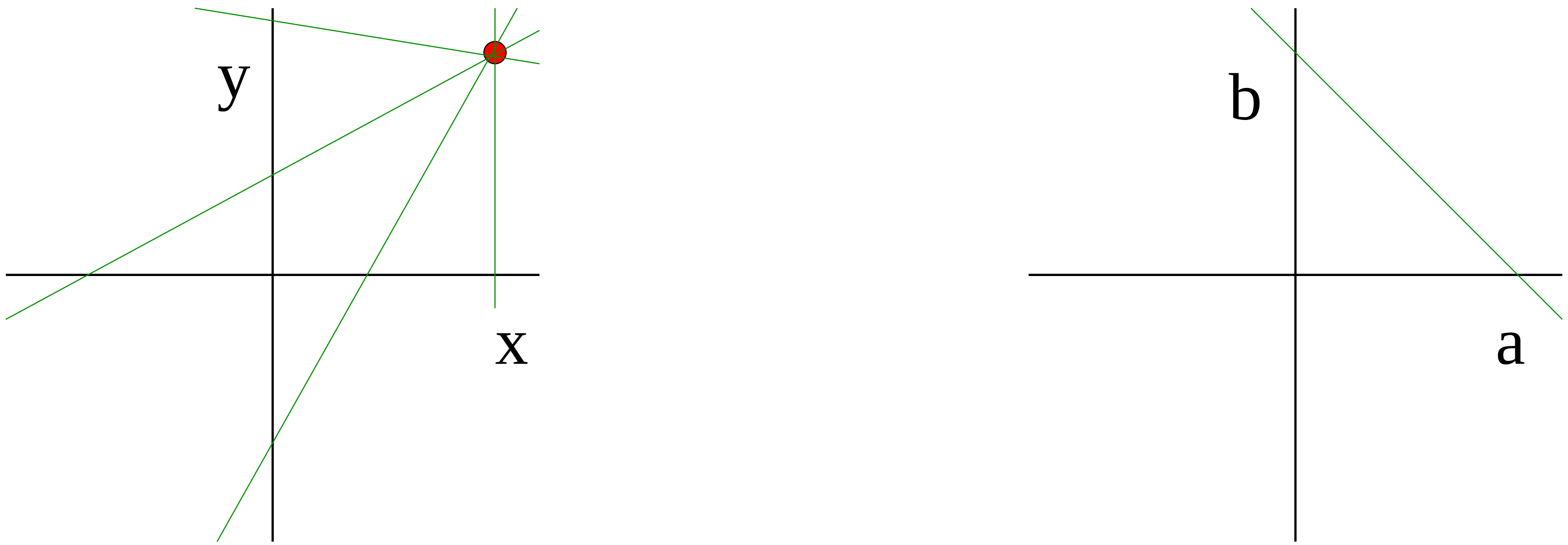
2. A linear regulator is of the form $U = \rho^T L^T L \rho$, where the matrix $L = 1$, D, D^2, *etc.*.
   SG, PL B597 (2004) 57.

3. Include known information into the Bayesian probability.
   G. P. Lepage *et al.*, *Nucl. Phys.*, B (Proc. Suppl.) 106 (2002) 12.

4. Fit a form with small parameters to the functional form, and accept or reject this hypothesis by the usual means.
   Pearson?

An example

Determine the parameters of the line $y = a + bx$ passing through (1,1)

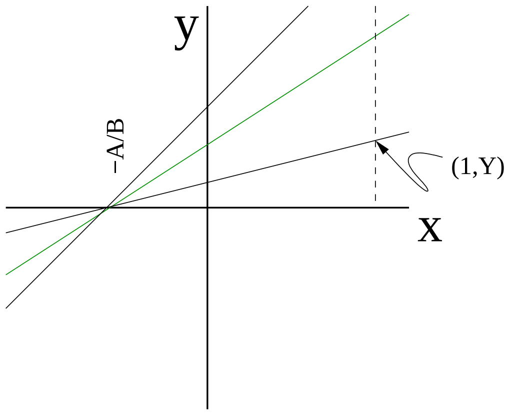


Simplified version of the actual problem to be solved: $2 \times L^3$ lattice.

Solution: method 1

Method 1: MEM

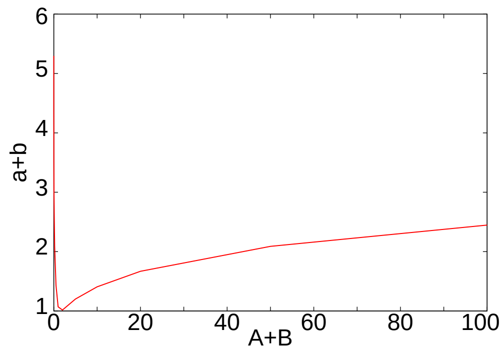
$$F(a,b) = (1-a-b)^2 + \beta\left(a\log\frac{a}{A} + b\log\frac{b}{B} - a - b\right)$$



The minimum is at

$$\frac{a}{A} = \frac{b}{B} = u \quad \text{where} \quad 1 - Yu = \frac{\beta}{2}\log u,$$

and $Y = A + B$. Solutions exist only for $Y > 0$.
If $Y < 1$ then $u > 1$ and vice versa.



The best fit does not pass through the data except when $A + B = 1$

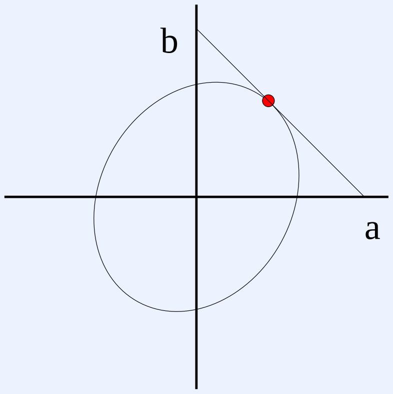
Solution: method 2

Method 2: General linear regulator $U = \rho^T L^T L \rho$

$$F(a,b) = (1-a-b)^2 + \beta(l_{11}a^2 + l_{22}b^2 + 2l_{12}ab)$$

U is positive definite. The minimum occurs at

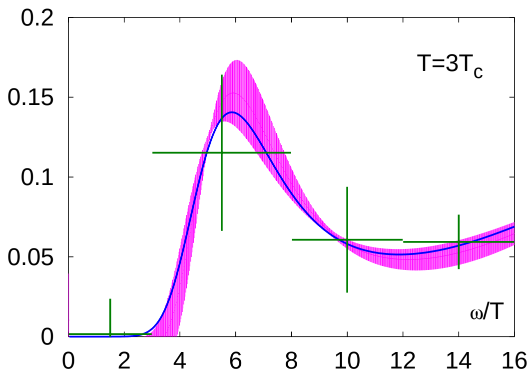
$$M \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{where} \quad M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \beta \begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix}$$

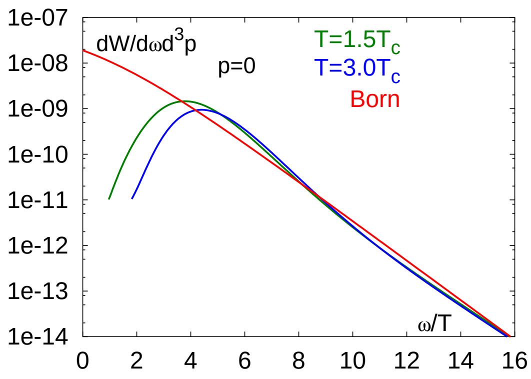


For all L the best fit passes through the data.

$$\text{Most probable } \beta = 0 : \quad \begin{pmatrix} a \\ b \end{pmatrix} = \overbrace{\frac{1}{1+x}\begin{pmatrix} x \\ 1 \end{pmatrix}}^{l_{11}\neq 0\ (x=l_{22}/l_{11})} \quad \text{or} \quad \overbrace{\frac{1}{1+x}\begin{pmatrix} 1 \\ x \end{pmatrix}}^{l_{22}\neq 0\ (x=l_{11}/l_{22})} \qquad (l_{12}=0).$$

Large ω using MEM





F. Karsch et al, Phys.Lett.B530:147,2002— Wilson quarks

Full agreement with Born for $\omega/T \geq 4$.

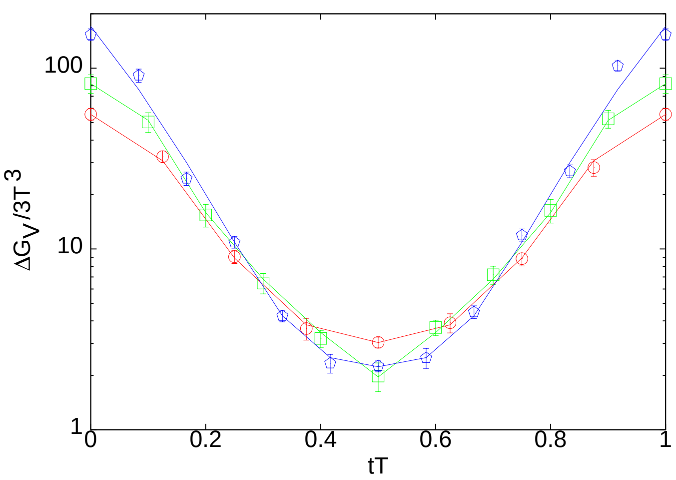
Default model: ideal gas behaviour. Output: $\rho(\omega)$ grows as ω^2 at large ω. Extracted value vanishes as $\omega \to 0$. Need to examine low ω region by another method in more detail.

Small ω using linear regulator

Since the problem is linear, work with

$$\Delta G(\omega, \mathbf{p}; T) = G_{full}(\omega, \mathbf{p}; T) - G_{ideal}(\omega, \mathbf{p}; T).$$

This gets rid of the ω^2 divergence at infinity, at the cost of the positivity of $\Delta\rho$. Use a linear regulator. This shows a bump at small ω. Second and higher bump at $\omega/T \approx$8–9.



SG, PL B597 (2004) 57.

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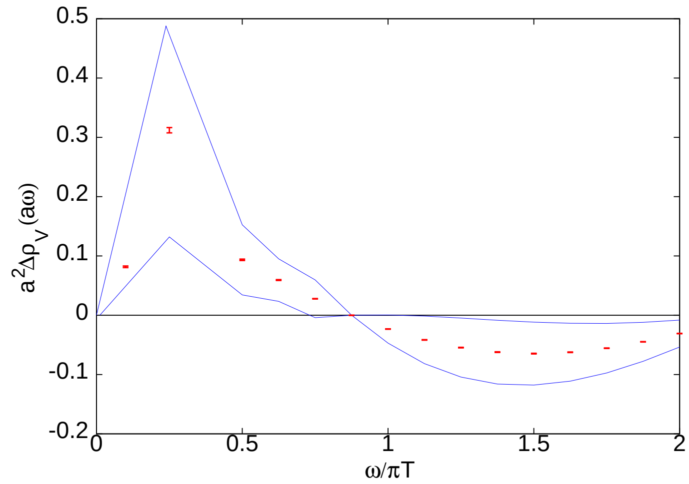

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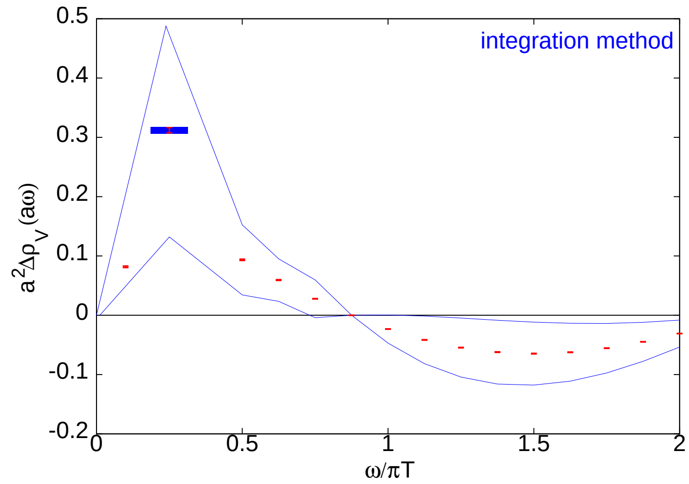

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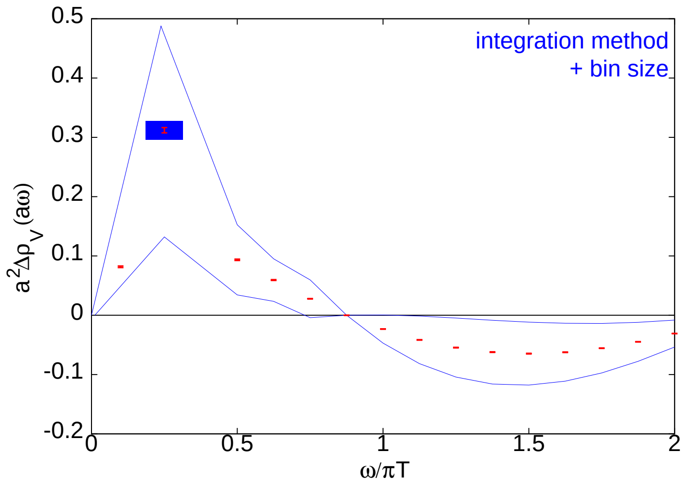

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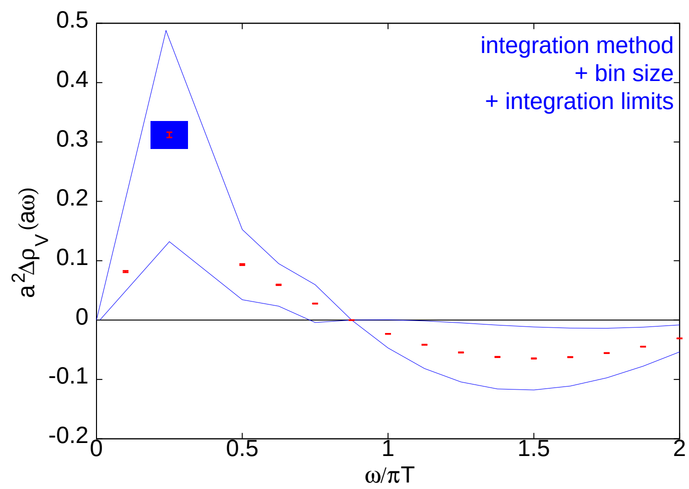

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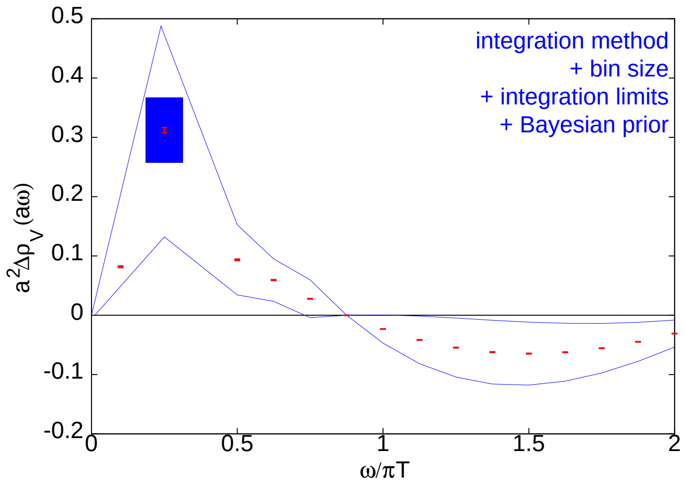

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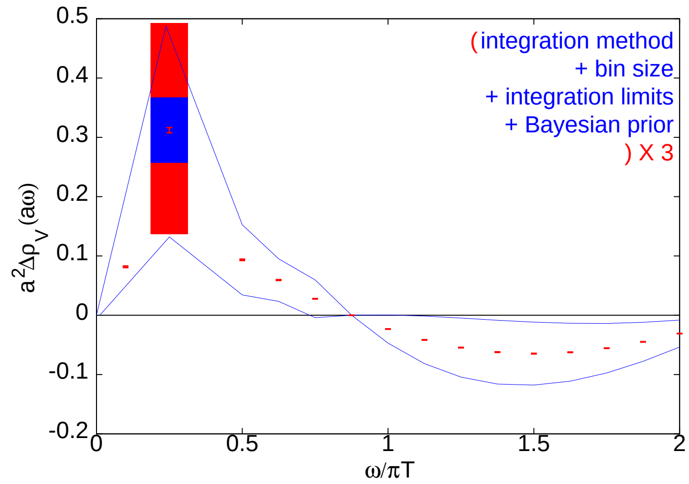

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Lattice gauge theory with parametrised Bayesian methods

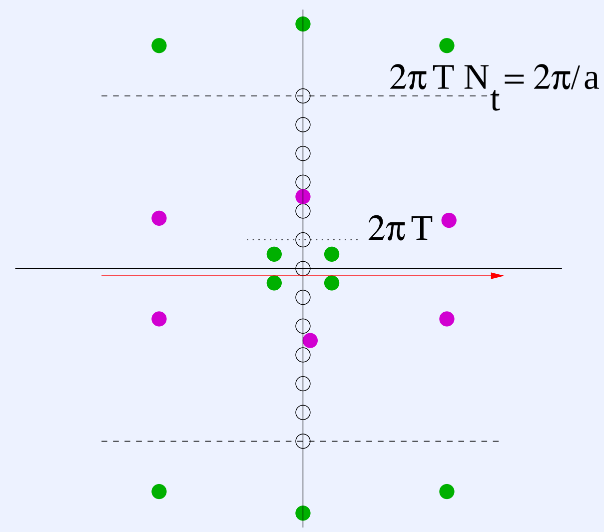
Use a sequence of parametrisations for the spectral density

$$\frac{\Delta\rho}{T^2} = \frac{z\sum_{n=0}^{N}\gamma_n z^{2n}}{1+\sum_{m=1}^{M}\delta_m z^{2m}}.$$

Use with Fourier space correlators—

$$\Delta G(\omega_n, \mathbf{p}; T) = \oint \frac{d\omega}{2i\pi}\,\frac{\Delta\rho(\omega, \mathbf{p}; T)}{\omega - \omega_n}$$

where $\omega_n = 2i\pi nT$.



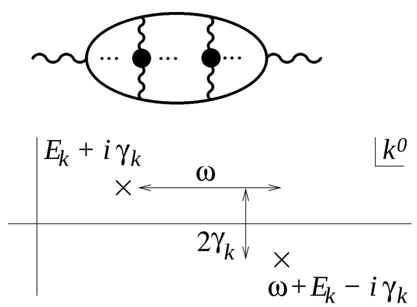
Two types of poles: S_2 is "relaxation time" and S_4 is "transport". S_2 ruled out.

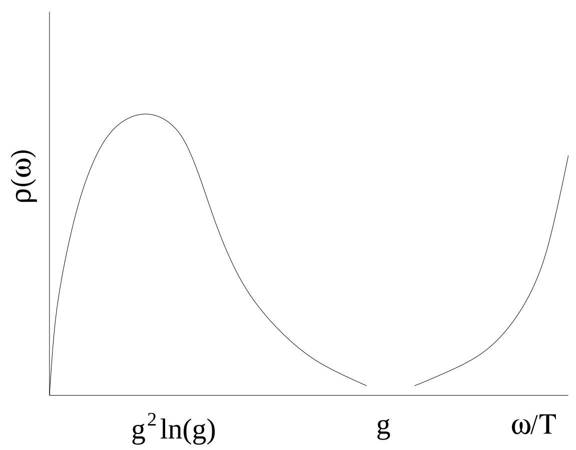
Use χ^2 parameter fitting if $N + M + 1 \leq N_t$, Bayesian otherwise.

F. Karsch and H. W. Wyld, *Phys. Rev.*, D 35 (1987) 2518; S. Sakai *et al.*, hep-lat/9810031

Pinch singularities and transport

There are pinch singularities at small external energy, ω, from ladder diagrams. These ladder diagrams correspond to multiple scatterings off particles in the plasma.





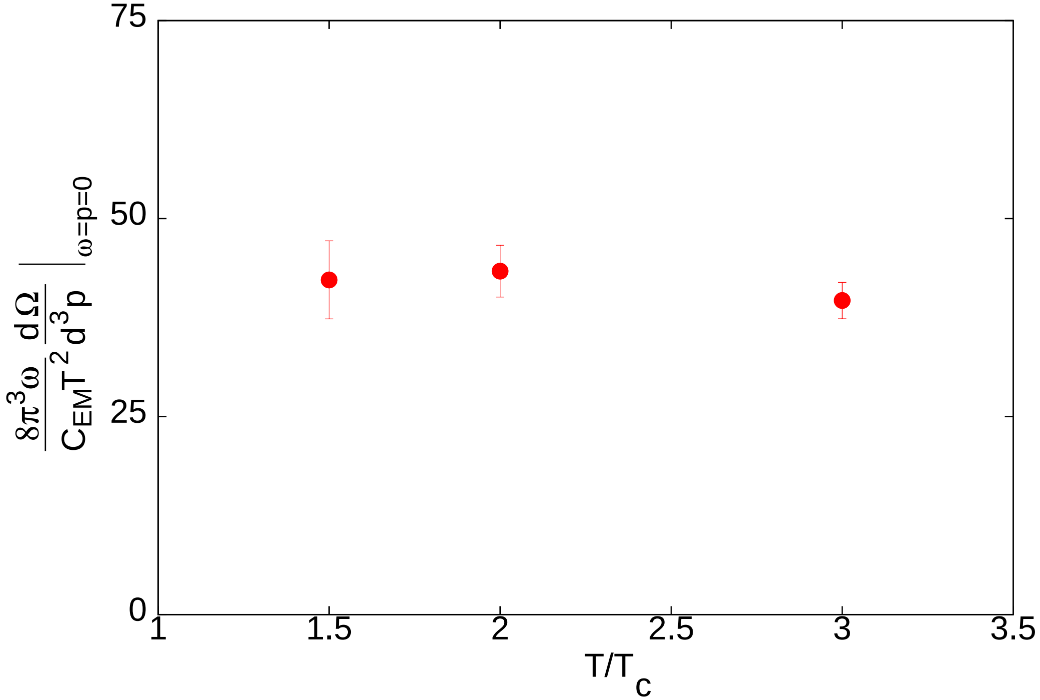
Transport: Arnold, Moore and Yaffe,

G. Aarts and J.M.M. Resco JHEP 0204:053,2002

Electrical conductivity: continuum limit

Electrical conductivity depends only on the parameter γ. Obtain this by marginalising over the remaining parameters. SG, PL B597 (2004) 57.

$\frac{\sigma}{T} \approx 7C_{EM}$ for $1.5 \leq T/T_c \leq 3$



Summary and phenomenology

1. Typical transport length/time scales in the plasma are $\sigma = C_{EM} n_q \tau_q / m$. Then $\tau_q \approx 0.2$ fm, hence $\tau_g \approx 0.1$ fm. Using similiar transport formula: $\eta/S \approx 0.2$.

2. Hydrodynamic description of the fireball works if its thermalisation time is less than 0.6 fm. A typical relaxation time in the plasma is τ_g and hence is short.

3. A soft photon mean free path is $\ell = \tau_q / C_{EM} \approx 4$ fm. Typical fireball dimensions at RHIC are 7 fm, so the fireball is marginally transparent to soft photons ($\omega \leq 200 MeV$). Small τ_g implies small ℓ.

4. Spontaneous thermal fluctuations of flavour even out by diffusion: $\sigma = \sum_f e_f^2 D_f \chi_f$, where D_f is a diffusion coefficient and χ_f is the particle number susceptibility. Therefore chemical signals visible only at freeze out.