The QCD phase diagram from the lattice

Sourendu Gupta

ILGTI: TIFR

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Zero baryon density
  Background
  Exact SU(2) flavour symmetry
  Exact SU(3) flavour symmetry
  Broken flavour symmetry

Finite Baryon Density
  The phase diagram
  Lattice simulations
  Summing the series

Reaching out to experiments
  Finding Gaussian fluctuations
  Testing QCD predictions

Summary
Outline

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Summary
How many flavours

- Flavour symmetries not exact: difference in masses of different flavours breaks symmetry. For example: if \( m_u = m_d \) then (within QCD) any two mutually orthogonal linear combinations of \( u \) and \( d \) are equivalent: symmetry. When \( m_u \neq m_d \), symmetry broken: these transformations change energy.

- Since \( m_{\pi^0} \simeq m_{\pi^\pm} \), flavour SU(2) is a good approximate symmetry of the hadron world. Flavour SU(3) is not useful without symmetry breaking terms (Gell-Mann and Nishijima).

- If some \( m \gg \Lambda_{QCD} \) then that quark is not approximately chiral. In QCD two flavours are light (\( m_{u,d} \ll \Lambda_{QCD} \)) and one is medium heavy (\( m_s \simeq \Lambda_{QCD} \)). Recall that \( m_\pi = 0.2m_\rho \) but \( m_K = 0.7m_\rho \). Limit \( m_\pi = 0 \): chiral symmetry.

- Do we have a two flavour phase diagram or a three flavour phase diagram, or something else?
The two flavour world

- What distinguishes the phases? Exact answer only for massless quarks. In the vacuum chiral symmetry is broken; \( \langle \bar{\psi} \psi \rangle \) **chiral condensate** is non-vanishing. Pions are the massless fluctuations around the vacuum. At high temperature \( \langle \bar{\psi} \psi \rangle = 0 \).

- When correlation lengths finite then susceptibility always finite:

\[
\chi = \int d^3 x C(x), \quad C(x) \approx \exp(-mx), \quad \xi = 1/m_{PS}.
\]

At critical points correlation lengths diverge; integral diverges, so susceptibility diverges.

- When quarks are massive, then at transition scalar mass degenerate with \( m_\pi \neq 0 \). No vanishing masses, so all susceptibilities finite. May still have a maximum as \( T \) changes: cross over for massive quarks.
The two flavour phase diagram

\[ T \]

2nd order chiral phase transition

line of cross over
no phase transition

\[ N_f = 2 \text{ QCD} \]

Pisarski and Wilczek, PR D 29, 338 (1984)
The two flavour phase diagram

\[ N_f = 2 \text{ QCD} \]

Pisarski and Wilczek, PR D 29, 338 (1984)
The three flavour world

- For three massless flavours $m_\pi = m_K = m_\eta = 0$. Chiral condensate distinguishes phases. However: first order phase transition; chiral condensate vanishes with a jump.

- If flavour symmetry exact ($m_\pi = m_K = m_\eta \neq 0$), then first order transition remains stable up to some point. Jump decreases continuously until it vanishes. This is a critical end point of this line.

- In the real world SU(3) flavour symmetry is broken ($m_\pi \neq m_K \neq m_\eta$). What is the phase diagram? Encoded in the Columbia plot.
The three flavour phase diagram

\[ \mu = 0 \quad \mu \neq 0 \]

**Experiment Summary**

- \( N_f = 2 \)
- \( N_f = 3 \)
- \( N_f = 2 + 1 \)

**Background**

The three flavour phase diagram

- Line of first order phase transitions
- 2nd order chiral end point

\( N_f = 3 \) QCD
The Columbia plot

Brown et al, PRL 65, 2491 (1990)
The Columbia plot

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The Columbia plot

\[ N_f = 2 \]

\[ N_f = 3 \]

\[ N_f = 2 + 1 \]

Brown et al, PRL 65, 2491 (1990)
Lattice results for the Columbia Plot

In $N_f = 2 + 1$:

$$m^{\text{crit}}_{\pi} \begin{cases} = 0.07 m_{\pi} & (N_t = 4) \\ < 0.12 m_{\pi} & (N_t = 6) \end{cases}$$

Endrodi et al, 0710.0988 (2007)

Similarly for $N_f = 3$.

Lattice results for $N_f = 2 + 1$

1. Two independent lattice computations (now) agree on the position of the crossover temperature for physical quark mass ($m_\pi \simeq 140$ MeV):

$$T_c \simeq 170 \text{ MeV}.$$ 

Aoki et al, hep-lat/0611014 (2006); HotQCD, 2010.

2. Clear evidence that susceptibilities do not diverge: no critical point, definitely a cross over. BW: difference between “chiral” and “deconfinement” cross overs. HotQCD: no such difference.

3. Chiral ($m_\pi = 0$) critical point: not yet well determined. Current studies indicate that expected divergences do occur. However, the approach to infinities are not yet completely under control. Ejiri et al, 0909.5122 (2009)
Lattice results for $N_f = 1 + 1$

No significant change in $T_c$ as $m_{\pi^0}/m_{\pi^\pm}$ is changed from 1 to 0.78 (physical value bracketed). Only one study; lattice spacings are coarse by today’s standards; finite size scaling yet to be performed.

Gavai, SG, hep-lat/0208019 (2002)

\[
\frac{T_c}{\Lambda_{\text{MS}}} = \begin{cases} 
0.49 \pm 0.02 & (m_{\pi^0}^2/m_{\pi^\pm}^2 = 1) \\
0.49 \pm 0.02 & (m_{\pi^0}^2/m_{\pi^\pm}^2 = 0.78)
\end{cases}
\]

Both results extrapolated to the physical value of $m_{\pi}/m_{\rho}$.
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Summary
The two flavour phase diagram

\[ \mu = 0 \quad \mu \neq 0 \]

Experiment Summary

Phase diagram

Lattice Series

The two flavour phase diagram

- Line of 2nd order phase transitions
- Tri-critical point
- Line of 1st order phase transitions

\[ T \]

\[ N_f = 2 \text{ QCD (chiral)} \]

The two flavour phase diagram

The two flavour phase diagram

The two flavour phase diagram

Lattice setup

Lattice simulations impossible at finite baryon density: **sign problem**. Basic algorithmic problem in all Monte Carlo simulations: no solution yet.

Bypass the problem; make a Taylor expansion of the pressure:

\[
P(T, \mu) = P(T) + \chi_B^{(2)}(T) \frac{\mu^2}{2!} + \chi_B^{(4)}(T) \frac{\mu^4}{4!} + \cdots
\]

Series expansion coefficients evaluated at \( \mu = 0 \).

Implies

\[
\chi_B^2(T, \mu) = \chi_B^{(2)}(T) + \chi_B^{(4)}(T) \frac{\mu^2}{2!} + \chi_B^{(6)}(T) \frac{\mu^4}{4!} + \cdots
\]

Series fails to converge at the critical point.

Series diverges

Radius of convergence of the series as a function of order $(a^{-1} = 1200 \text{ MeV})$

Gavai, SG, 0806.2233 (2008)
Series diverges

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Radius of convergence of the series as a function of order \( (a^{-1} = 1200 \text{ MeV}) \)

Gavai, SG, 0806.2233 (2008)
Dependence on quark mass

\[ a^{-1} = 800 \text{ MeV} \]

SG, hep-lat/0608022 (2006)
Systematic effects

1. Series expansion carried out to 8th order. What happens when order is increased? Intimately related to finite volume effects. Finite size scaling tested; works well


2. What happens when strange quark is unquenched (keeping the same action)? Numerical effects on ratios of susceptibility marginal when unquenching light quarks

   Gavai, SG, hep-lat/0510044; see also RBRC 2009; de Forcrand, Philipsen, 2007, 2009

3. What happens when $m_\pi$ is decreased? Estimate of $\mu_B^E$ may decrease somewhat: first estimates in


4. What happens in the continuum limit? Estimate of $\mu_B^E$ may increase somewhat

   Gavai, SG 2008; SG 2009
The critical point of QCD

$$\mu = 0 \quad \mu \neq 0$$

**Experiment Summary**

- **Phase diagram**
  - **Lattice**
  - **Series**

The critical point of QCD

\[
\frac{\mu^E}{T^E} \sim \begin{cases} 
1.8 \pm 0.1 & N_f = 2, \frac{1}{a} = 1200 \text{ MeV} \\
1.5 \pm 0.4 & N_f = 2 + 1, \frac{1}{a} = 800 \text{ MeV}
\end{cases}
\]

Gavai, SG, 0806.2233 (2008)

BNL-Bielefeld-GSI, unpublished, 2010

Comparable \( m_\pi \); normalized to same estimator.
The critical point of QCD

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**Experiment Summary**

**Phase diagram**

**Lattice Series**

The critical point of QCD

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\]

comparable \( m_\pi \); normalized to same estimator.
Extrapolating physical quantities

\[ \Delta p = \chi_B^{(2)} \frac{\mu^2}{2!} + \chi_B^{(4)} \frac{\mu^4}{4!} + \cdots - \chi_S^{(2)} \frac{\mu_S^2}{2!} - \cdots \]

MILC Collaboration, 1003.5682 (2010)
Critical divergence: summation bad, resummation good

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Summary
Locating the critical end point in experiment

Measure the (divergent) width of momentum distributions


Better idea, use conserved charges, because at any normal (non-critical) point in the phase diagram:

\[ P(\Delta B) = \exp \left( -\frac{(\Delta B)^2}{2VT\chi_B} \right). \quad \Delta B = B - \langle B \rangle. \]

At any non-critical point the appropriate correlation length (\(\xi\)) is finite. If the number of independently fluctuating volumes (\(N = V/\xi^3\)) is large enough, then net \(B\) has Gaussian distribution: **central limit theorem**

Landau and Lifschitz

Bias-free measurement possible

Is the top RHIC energy non-critical?

Check whether CLT holds.

Recall the scalings of extensive quantity such as $B$ and its variance $\sigma^2$, skewness, $S$, and Kurtosis, $K$, given by

$$B(V) \propto V, \quad \sigma^2(V) \propto V, \quad S(V) \propto \frac{1}{\sqrt{V}}, \quad K(V) \propto \frac{1}{V}. $$

Coefficients depend on $T$ and $\mu$. So make sure that the nature of the physical system does not change while changing the volume.

This is a check that microscopic physics is forgotten (except two particle correlations).
STAR measurements

Perfect CLT scaling: remember only $VT\chi_B$? or some other physics?

Can we recover microscopic physics?

Can we test QCD?
What to compare with QCD

The cumulants of the distribution are related to Taylor coefficients—

\[
[B^2] = T^3 V \left( \frac{\chi^{(2)}}{T^2} \right), \quad [B^3] = T^3 V \left( \frac{\chi^{(3)}}{T} \right), \quad [B^4] = T^3 V \chi^{(4)}.
\]

\(T\) and \(V\) are unknown, so direct measurement of QNS not possible (yet). Define variance \(\sigma^2 = [B^2]\), skew \(S = [B^3]/\sigma^3\) and Kurtosis, \(\mathcal{K} = [B^4]/\sigma^4\). Construct the ratios

\[
m_1 = S\sigma = \frac{[B^3]}{[B^2]}, \quad m_2 = \mathcal{K}\sigma^2 = \frac{[B^4]}{[B^2]}, \quad m_3 = \frac{\mathcal{K}\sigma}{S} = \frac{[B^4]}{[B^3]}.
\]

These are comparable with QCD provided all other fluctuations removed.

SG, 0909.4630 (2009)
How to compare with QCD

Since two parameters in QCD computations ($T$ and $\mu$), possible measurements lie on a surface. Make three independent measurements (also independent of fireball volume) to check this. Agreement of QCD and data implies control over other sources of fluctuations.

If in equilibrium then CP implies non-monotonic behaviour of $m_1 = S \sigma$, etc. with energy. Near CP system drops out of equilibrium: finite lifetime and finite size. Lack of agreement with QCD is signal of CP!
Surprising agreement with lattice QCD: implies non-thermal sources of fluctuations are very small; temperature does not vary across the freezeout surface.

Gavai, SG, 1001.3796 (2010)

Lattice prediction:

\[ \kappa \sigma^2 = 0.88 \pm 0.0 \]
How to find the critical point: be lucky!
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![Diagram showing the freezeout curve and the coexistence curve in a phase diagram with axes $T/T_c$ and $\mu_B/T$. The critical point is marked by a black dot where the two curves intersect.]
How to find the critical point: be lucky!

$T/T_c$ vs. $\mu_B/T$ with the coexistence curve and freezeout curve.
Lattice predictions along the freezeout curve

Filled boxes: $a = 1/(4T)$, unfilled circles: $a = 1/(6T)$.

Gavai, SG, 1001.3796 (2010)
Lattice predictions along the freezeout curve

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Summary
Main results

- The strange quark is heavy; light quarks determine the shape of the phase diagram. The cross over temperature now under control: $T_c \simeq 170$ MeV. SU(2) flavour symmetry breaking unlikely to change $T_c$.

- Approach to zero quark mass is beginning to come under control for the first time. Consistency with continuum symmetry arguments may be established. (Computation very hard with small pion mass)

- Lattice determines series expansion of pressure; indicates a critical point in QCD. Range of predictions: $\mu^E / T^E \simeq 1.5–2.5$. Physical quantities can be found by resumming the series expansion (e.g., Padé approximants).

- Extrapolation of lattice results to the experimentally known freezeout curve possible. First results in surprising agreement with experiment. Need to check CLT and determine ratios of moments.