Controlling errors in the continuation of lattice results to finite chemical potential

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QCD at finite chemical potential

- Action has indefinite sign, so direct simulation is not possible.
- Thermodynamic quantities, pressure, entropy density, number density are real.

  Hasenfratz, Karsch 1983; Bilic, Gavai 1983
- May be able to obtain some information using series expansions and resummations. Maclaurin expansion:

$$\frac{1}{T^4}P(T,\mu_B) = \frac{P(T)}{T^4} + \frac{1}{2}\,\frac{\chi^{(2)}(T)}{T^2}\left(\frac{\mu_B}{T}\right)^2 + \frac{1}{4!}\,\chi^{(4)}(T)\left(\frac{\mu_B}{T}\right)^4 + \cdots$$

  Gavai and SG, 2002
- Breakdown of series expansion most easily studied; use successive estimators of radius of convergence.

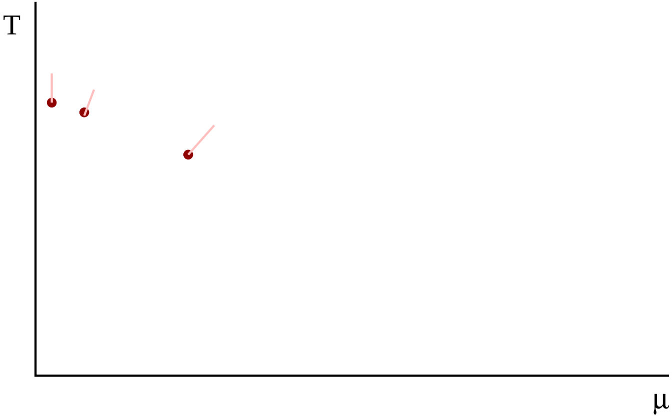
  Gavai and SG, 2005, 2008, 2012

Experimental data

Maybe possible to determine thermodynamic state variables, *i.e.*, reach equilibrium at “freeze out”. Interactions negligible after freeze out so ideal gas may be good description.

Braun-Munzinger, Stachel, Cleymans, Redlich, Becattini

Experimental data



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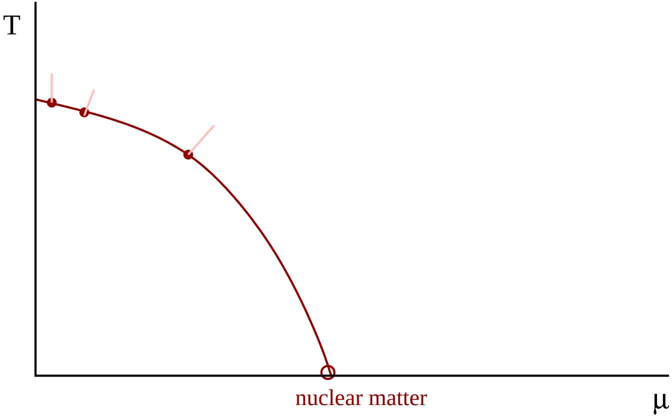

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Fluctuations of conserved quantities

- In a single heavy-ion collision, each conserved quantity (B, Q, S) is exactly constant when the full fireball is observed. In a small part of the fireball they fluctuate: from part to part and event to event.

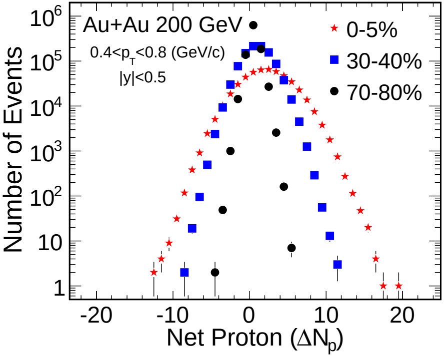
  Asakawa, Heinz, Muller 2000; Jeon, Koch 2000

- If $\xi^3 \ll V_{obs} \ll V_{fireball}$, then fluctuations can be explained in the grand canonical ensemble: energy and B, Q, S allowed to fluctuate in one part by exchange with rest of fireball: diffusion. Theoretical control requires transport coefficients.

  SG 2007

- Is $V_{obs} \ll V_{fireball}$? Experimental checks needed; corrections possible. Is $V_{obs} \gg \xi^3$? Measure screening correlators and use finite size scaling. Do fluctuations freeze out at the same temperature as hadron chemistry? Peclet number: depends on transport coefficients.

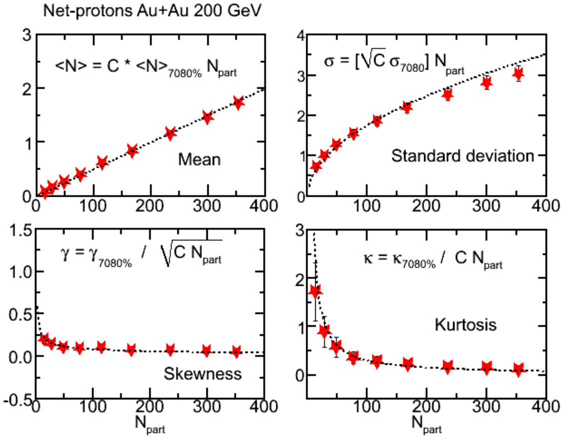
Event-to-event fluctuations



STAR arxiv:1004.4959

Central rapidity slice taken. p_T of 400–800 MeV. Important to check dependence on impact parameter. Protons observed: isospin fluctuations small.

Shape of distribution



STAR: QM 2009

Shape of distribution captured in cumulants $[B^n]$. Cumulants change with volume (proxy: N_{part}), by central limit theorem.

QCD predictions needed at finite μ_B

Shape variables: $[B^n] = (VT^3)T^{n-4}\chi_B^{(n)}(T,\mu)$. Ratios of cumulants are thermodynamic state variables:

$$m_0: \quad \frac{[B^2]}{[B]} = \frac{T\chi_B^{(2)}}{\chi_B^{(1)}}$$

$$m_1: \quad \frac{[B^3]}{[B^2]} = \frac{T\chi_B^{(3)}}{\chi_B^{(2)}} = S\sigma$$

$$m_2: \quad \frac{[B^4]}{[B^2]} = \frac{T^2\chi_B^{(4)}}{\chi_B^{(2)}} = \kappa\sigma^2$$

$$m_3: \quad \frac{[B^4]}{[B^3]} = \frac{T\chi_B^{(4)}}{\chi_B^{(3)}} = \frac{\kappa\sigma}{S}$$

SG, 2009; Athanasiou, Rajagopal, Stephanov, 2010

The need for Padé approximants

Can we sum these series?

$$\frac{1}{T^4}P(T,\mu_B) = \frac{P(T)}{T^4} + \frac{1}{2}\,\frac{\chi^{(2)}(T)}{T^2}\left(\frac{\mu_B}{T}\right)^2 + \frac{1}{4!}\,\chi^{(4)}(T)\left(\frac{\mu_B}{T}\right)^4 + \cdots$$

$$\frac{\chi^{(2)}(T,\mu_B)}{T^2} = \frac{\chi^{(2)}(T)}{T^2} + \frac{1}{2!}\,\chi^{(4)}(T)\left(\frac{\mu_B}{T}\right)^2 + \frac{1}{4!}\,T^2\chi^{(6)}(T)\left(\frac{\mu_B}{T}\right)^4 + \cdots$$

At very small μ_B series expansion may be useful (LHC conditions).

But series may diverge at or near the freeze out curve, so truncated series expansion may be wrong. The shape variables m_i have simple poles at a critical point. So useful to try Padé approximants.

Need to understand error propagation.

Gavai, SG 2010

The problem?

Want to evaluate the $[0,1]$ Padé approximant

$$P(z;a) = \frac{1}{z-a},$$

at various $z = \mu_B/T$ for a determined from lattice measurement.
If a has Gaussian errors, then for any z, there is a probability that $a = z$. So the mean and variance of P both diverge.
See this another way. Assume that the distribution of a is Gaussian with mean 1 and variance σ^2. Then the distribution of P at fixed z is given by

$$p(P;z) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\frac{1}{P^2}\mathrm{e}^{-(z-1-1/P)^2/(2\sigma^2)}.$$

The distribution is normalizable but none of the moments exist.

Is there a meaningful regularization?

Yes. Because of finite statistics the maximum and minimum values of the Padé approximant are always bounded.

If one estimates $P(z; a)$ by a bootstrap, then one should take the number of bootstrap samples to be $\mathcal{O}(N)$. By accounting for the restricted range $|P| \leq \Lambda$, all the integrals are regularized. If the measurements are made with statistics of N, then $\sigma^2 \propto 1/N$. If

$$\epsilon(\Lambda) = 1 - \int_{-\Lambda}^{\Lambda} dP p(P; z),$$

and Λ is chosen such that $N\epsilon(\Lambda) \ll 1$, then the regularization is sensible.

With $\sigma^2 \propto 1/N$ and $\epsilon \propto 1/N$, in the limit $N \to \infty$ is it possible to remove the regularization and have finite $\langle P \rangle$ and $\langle P^2 \rangle$?

Finite results

With increasing N one can arrange $N\epsilon$ to be constant by scaling $\Lambda \to \zeta\Lambda$ with $\zeta \propto N^{3/2}$. For Gaussian distributed a,

$$\delta\langle P\rangle \simeq \mathrm{e}^{-K(1-z)^2 N} \log(\zeta/\zeta')$$

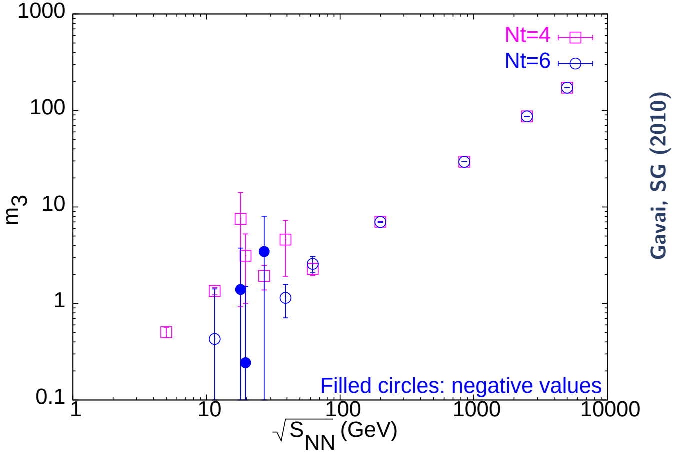
$$\delta\langle P^2\rangle \simeq \mathrm{e}^{-K(1-z)^2 N}(\zeta-\zeta')\Lambda\sigma$$

As a result a bootstrap estimation will lead to bounded mean and error for the Padé approximant except close to the pole.

Beyond the Gaussian approximation: bound the growth of $\langle P\rangle$ and $\langle P^2\rangle$ by verifying that the estimate of the error in the pole narrows faster than the growth of the probability in the tail of the distribution of the value of $P(z;a)$.

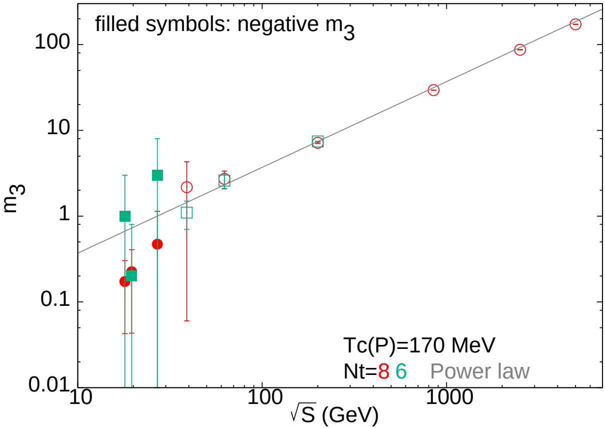
Numerical experiments work when a is the ratio of two Gaussian distributed variates (each with variance going as $1/N$).

Predictions along the freezeout curve



Lattice predictions along the freezeout curve of HRG models using $T_c = 170$ MeV.

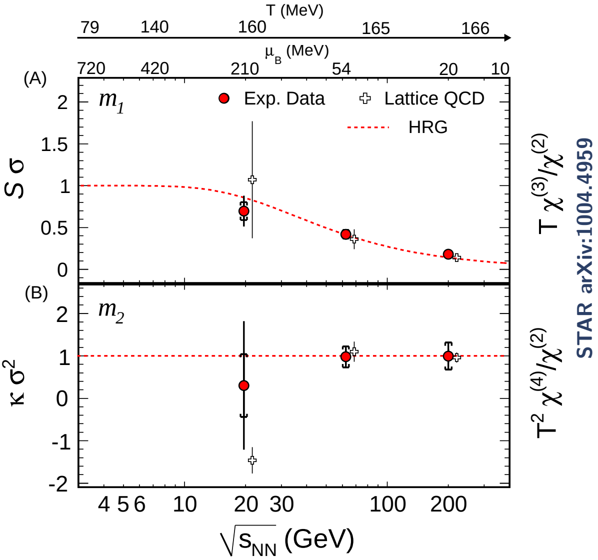
Smaller lattice spacing



Datta, Gavai, SG (2012)

Lattice predictions along the freezeout curve of HRG models using $T_c = 170$ MeV.

Checking the match



Further developments

1. Independent evidence of thermalization at freezeout. Rough at present because of errors in the experiment and lattice. Refinements required to test this critically: if it fails then may be QCD matter is not that opaque after all.

   Gavai SG, 2010; GLMRX, 2011

2. By leaving the lattice scale unspecified, can use a comparison of lattice and experiment to give a scale. Eventually add to the repertoire of scale settings possible for lattice.

   Gavai SG, 2010; GLMRX, 2011

3. If the scale setting is done as usual using $T = 0$ properties of hadrons, then one can extract freezeout parameters using finite μ_B extrapolation of lattice measurements. Useful if done with care (using resummed series).

   Gavai SG, 2010; Karsch 2012

4. Most exciting: at some beam energy thermalization may not be seen. Then understand why. Critical point, or something else?