Borel sum

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Borel summation

Feynman diagram expansions are typically asymptotic series expansion.

⊙ If $f(x)$ is a function and $R_n(x)$ some fn with integer index n which is divergent for $n \to \infty$ for any non-zero x. (zero radius of convergence)

Then, $f(x) \sim R(x)$ [asymptotically equivalent] if

$$\lim_{x\to\infty} x^n \left[f(x) - R_n(x) \right] = 0 \qquad \text{for finite } n.$$

ie $f(x)$ can be made arbitrarily close to $R_n(x)$ for any n by making x arbitrarily large (or small depending on convention).

On the otherotherhand

$$\lim_{n\to\infty} x^n \left[f(x) - R_n(x) \right] \to \infty$$

means $R_n(x)$ has zero radius of convergence.

$R_n(x)$ is the asymptotic series for $f(x)$. A property of $R_n(x)$ is that for a given x, the $R_n(x)$ converges towards a number for $n \simeq m$ and it then again starts diverging.

- a convergent series need not be asymptotic

$$\lim_{x\to\infty} x^n \left[e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right] \to \infty$$

Example: $R_n(x) = n!\, x^n$ has zero radius of convergence.

$$\text{But,} \quad \sum_0^\infty n!\, x^n = \sum_0^\infty \int_0^\infty dt \cdot t^n \cdot e^{-t} \cdot x^n$$

this step is not Kosher, but this is one way of defining Borel sum.

$$= \int_0^\infty dt\, e^{-t} \sum_{n=0}^\infty (tx)^n$$

$$= \int_0^\infty dt\, \frac{e^{-t}}{1-xt} = f(x) \quad \text{is convergent for } x \leq 0$$

Therefore $f(x) \sim R(x)$ for $x < 0$.

[A way to define Borel sum: for any divergent series

let $R(x) = \sum_n a_n x^n$ is divergent.

$$\text{Then,} \quad R(x) = \sum_n n!\, \frac{a_n x^n}{n!} = \sum_n \int_0^\infty dt\, t^n e^{-t}\, \frac{a_n x^n}{n!}$$

$$= \int_0^\infty dt\, e^{-t} \underbrace{\sum_n \frac{a_n (xt)^n}{n!}}_{}$$

If this converges, then Borel sum defined.

Why important: Usually perturbative expansion (Feynman graph) is an asymptotic series. A meaningful answer is extracted by Borel Sum.