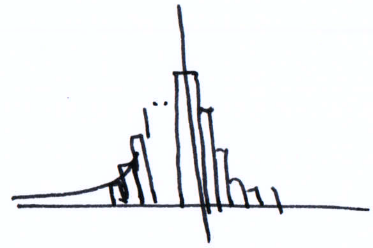
Stat mech is a probilistic description of world

basic concepts that we shall need!

Let x is a random variable. $x \in Re$.



~~p(x) prob. density~~

- If x is discrete, then

$$p(x) := \text{Prob of } x.$$

- If x is continuous

$$p(x)\, dx := \text{Prob for } (x, x+dx).$$

~~Moment generating function.~~

- Moments

$$\langle x^n \rangle := n\text{th moment.}$$

- moment generating function ($\lambda \in Re$)

$$g(\lambda) = \langle e^{\lambda x} \rangle = 1 + \lambda \langle x \rangle + \frac{\lambda^2}{2!} \langle x^2 \rangle + \frac{\lambda^3}{3!} \langle x^3 \rangle + \cdots$$

$$\Rightarrow \langle x^n \rangle = \frac{d^n g}{d\lambda^n}\Big|_{\lambda=0}$$

- ~~Measure of fluctuation~~
- Cumulants: a measure of fluctuations

$$\langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - 2 \langle x \rangle \langle x \rangle + \langle x \rangle^2$$

$$= \underbrace{\langle x^2 \rangle - \langle x \rangle^2}_{\text{variance}} = \langle x^2 \rangle_c$$

variance $\equiv$ second cumulant.

Cumulants are defined by cumulant generating function.

$$\mu(\lambda) = \log \langle e^{\lambda x} \rangle = \log \left[1 + \lambda \langle x \rangle + \frac{\lambda^2}{2!} \langle x^2 \rangle + \cdots \right]$$

$$= \lambda \langle x \rangle + \frac{\lambda^2}{2!} \left[\langle x^2 \rangle - \langle x \rangle^2 \right] + \frac{\lambda^3}{3!} \langle x^3 \rangle_c + \cdots$$

$$\Rightarrow \langle x^n \rangle_c = \mu^{(n)}(\lambda)\Big|_{\lambda=0}$$

[like cancelling open diagrams, and only taking connected diagrams]

[see wiki]

Remark. Sometimes $g(\lambda)$ for reals does not exist, but for $\lambda = ik$ it exist.

$$\hat{g}(k) = \langle e^{ikx} \rangle$$

characteristic function.

Remark. for Gaussian distribution (normal distribution)

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\langle x\rangle)^2}{2\sigma^2}} \qquad \sigma^2 = \langle x^2\rangle - \langle x\rangle^2.$$

all moments are non-zero, but only first and second cumulants are non-zero.

$$\mu(\lambda) = \lambda\langle x\rangle + \frac{\lambda^2}{2}\cdot\sigma^2$$

Remark. Cumulants are NOT centered moments

$$\langle x^3\rangle_c \neq \langle (x-\langle x\rangle)^3\rangle$$

Remark. For indep variables x, y

$$\langle e^{\lambda(x+y)}\rangle = \langle e^{\lambda x}\rangle\langle e^{\lambda y}\rangle \Leftarrow \text{because } p(x,y) = p(x)\,p(y).$$

- What we shall cover?
  - Central limit theorem, and when it fails.
    How does that lead to super/sub diffusion?
  - stable distributions as fixed points.

Later:

Basic concepts in probability

⊛ See the note of Abhishek. Dhar.
or note of Dauchot & Démery.

* statistical physics of non-eq systems is largely based on stochastic processes.

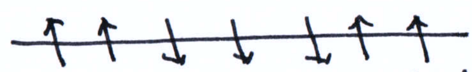
Important limit laws in probability:

Ref. Interesting examples. see exercises in 1st and 2nd chapter of book of Sethna.

Motivation. In physics we often ask how microscopic constituents "add up" to give a macroscopic observable.

Ex 1: net magnetization in a spin system

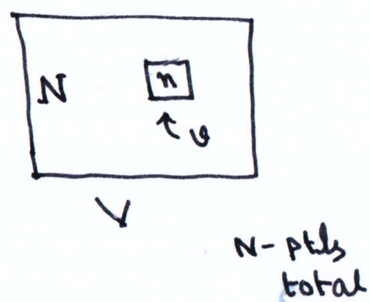
$$M = \sum_i S_i$$



Ex 2: number of particle in small box in a room

$$n \equiv M = \sum_i S_i$$

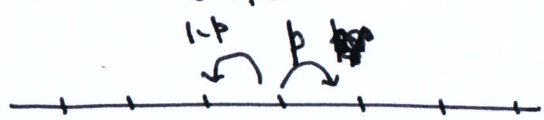
$S_i = \begin{cases} 0 \\ 1 \end{cases}$ if the ith ptl is inside the box v



Ex 3: Random walk position after N steps

$$X_N \equiv M = \sum_i S_i$$

$S_i = \pm 1$ for right/left jump



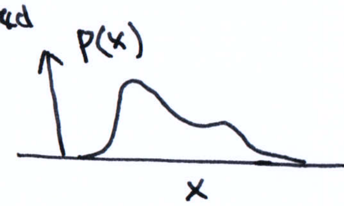
Ex 4: Emperical mean in an experiment.

Q. What are the limiting distribution for sums of random variables?

law of large numbers, Central limit theorem, stable distribution, Diffusion and anomalous diffusion, ...

For
* Clean mathematical answer, take i.i.d random variables X

identical indendent distributed



Q. What is the limiting distribution of

$$M_N = \sum_{i=1}^{N} x_i \qquad \text{when } N \text{ is large.}$$

[Note. when iid is justified? think finite correlation length/time]

Answer for Mean: law of large numbers.

Sample average $\frac{1}{N} M_N \xrightarrow[N\to\infty]{} \langle x\rangle$ ← if it exist $\left[\text{exception } P(x) = \frac{1}{x^2+1}\right]$

Mathematically (convergence in probabilistic sense)

$$\boxed{\lim_{N\to\infty} \mathrm{Prob}\left[\left|\frac{M_N}{N} - \langle x\rangle\right| > \epsilon\right] = 0 \text{ for any } \epsilon > 0.}$$

Don't confuse with $\langle M_N\rangle = \sum_i \langle x_i\rangle = N\langle x\rangle$

N not need to be large.

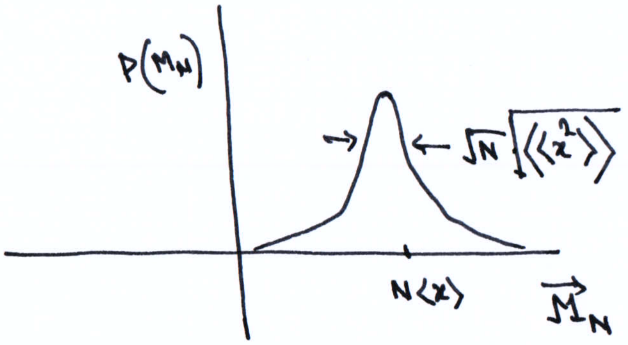
Answer for fluctuations:

$$\langle M_N^2\rangle = \sum_i \sum_j \langle x_i x_j\rangle = \sum_i \langle x_i^2\rangle + 2\sum_i \sum_{j>i} \overbrace{\langle x_i\rangle\langle x_j\rangle}^{\text{indep distr.}}$$

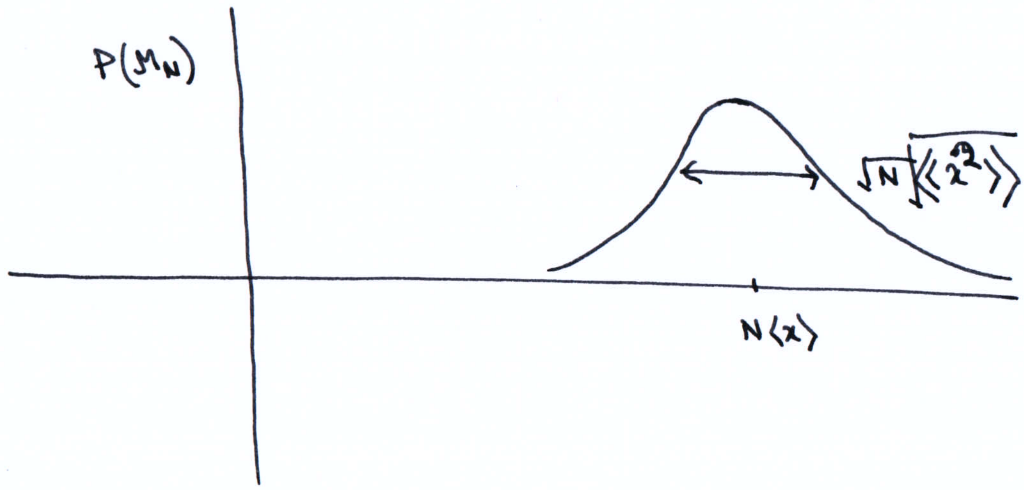
$$= N\langle x^2\rangle + N(N-1)\langle x\rangle^2$$

$$= N[\langle x^2\rangle - \langle x\rangle^2] + (N\langle x\rangle)^2$$

$$\langle M_N^2\rangle - \langle M_N\rangle^2 = N[\langle x^2\rangle - \langle x\rangle^2]$$



$\langle\langle x^2\rangle\rangle \equiv \langle x^2\rangle_c$



Typical fluctuations $\sim \sqrt{N}$

Q. What about the distribution of M_N?

- Central limit theorem:

$$P(M_N) \simeq \frac{1}{\sqrt{2\pi\langle\langle M_N^2\rangle\rangle}}\, e^{-\frac{(M_N-\langle M_N\rangle)^2}{2\langle\langle M_N^2\rangle\rangle}} \qquad \text{for large } N$$

$$= \frac{1}{\sqrt{2\pi N\langle\langle x^2\rangle\rangle}}\, e^{-\frac{(M_N-N\langle x\rangle)^2}{2N\langle\langle x^2\rangle\rangle}}$$

"no-matter" what the distr. of x

Mathematically.

$$P\left(M_N = N\langle x\rangle + \sqrt{N}\cdot z\right) \xrightarrow[N\to\infty]{} \frac{1}{\sqrt{2\pi\langle\langle x^2\rangle\rangle}}\, e^{-\frac{z^2}{2\langle\langle x^2\rangle\rangle}}$$

$\Rightarrow$

Proof : for a standard proof see Sanjib's lecture note, pag 10.

One intuitive way.

Check $\Longrightarrow$ • Will use that for Gaussian, all cumulants above 2 are zero.

$e^{\lambda\langle x\rangle + \frac{\lambda^2}{2}\langle x^2\rangle_c}$ show $\swarrow$ • $\langle e^{\lambda M}\rangle = \langle e^{\lambda \sum_i x_i}\rangle \underset{\text{iid.}}{=} \prod_{i=1}^{N} \langle e^{\lambda x_i}\rangle = \langle e^{\lambda x}\rangle^N$

$$\Rightarrow \log\langle e^{\lambda M}\rangle = N \log\langle e^{\lambda x}\rangle$$

$$\Rightarrow \log\left\langle e^{\lambda[M - N\langle x\rangle]}\right\rangle = N \log\left\langle e^{\lambda[x-\langle x\rangle]}\right\rangle$$

By definition

$$\lambda \underbrace{\langle[M-N\langle x\rangle]\rangle_c}_{0} + \frac{\lambda^2}{2}\langle[\quad]^2\rangle_c + \frac{\lambda^3}{3}\langle[\quad]^3\rangle_c + \cdots = N\left\{\langle[x-\langle x\rangle]\rangle_c \lambda + \frac{\lambda^2}{2}\langle[\,]^2\rangle_c + \cdots\right\}$$

$$\Rightarrow \langle[M-N\langle x\rangle]^2\rangle_c = N\langle[x-\langle x\rangle]^2\rangle_c$$

$$\langle[\quad]^3\rangle_c = N\langle[\,]^3\rangle_c$$

$$\vdots \qquad\qquad \vdots$$

This means

$$\left\langle\left(\frac{[M-N\langle x\rangle]}{\sqrt{N}}\right)^2\right\rangle_c = \langle(x-\langle x\rangle)^2\rangle_c$$

$$\langle(\quad)^3\rangle_c = \frac{1}{\sqrt{N}} \cancel{*} \longrightarrow 0 \quad \text{for } N\to\infty$$

all higher cumulants vanish.

$\Rightarrow$ for $\quad \dfrac{M_N - N\langle x\rangle}{\sqrt{N}} =$ [redacted] $\cdot z$

$$\langle e^{\lambda z}\rangle = \underbrace{\langle (x-\langle x\rangle)^2\rangle}_{\langle x^2\rangle_c} \cdot \frac{\lambda^2}{2} \qquad \text{for } N \to \infty$$

$$\Rightarrow \quad \mathrm{Prob}(z) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{z^2}{2\sigma^2}}$$

[Q. what happens if we look at fluctuations larger than $\sqrt{N}$, ie beyond typical. We will come back to this. (large deviations).]

<u>Remark</u>: convergence to Gaussian distribution could also occur for correlated random variables, [redacted] and for non-identical variables under reasonable conditions.

[Naively, if contribution of individual x_i to net variance $\langle M^2\rangle$ is small for large N, and if correlations are not long ranged].

<u>Exercise</u>. ① Show that sums of correlated Gaussian random variables is Gaussian. ? [Tutorial 1]

② What happens for $M = \sum_i a_i x_i$?

* <u>Term paper</u>: Get central limit theorem using RG.

[search article by Jona-Lasinio]

* CLT is perhaps the simplest example of universality, and many stat-phys questions can be phrased as generalization of CLT.