Fluctuating hydrodynamics for system with

(Many degrees of freedom)

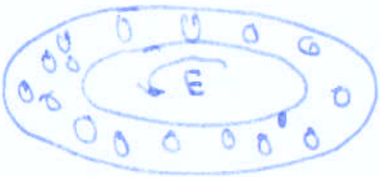
⊙ Refs: ① Kipnis and Landim: Scaling limits of interacting particle systems. Springer publication (2013).

② H. Spohn (1991): Large scale dynamics of interacting particles Springer.

③ Eyink, Lebowitz, Spohn (1996): Hydrodynamics and fluctuations outside of local equilibrium: driven diffusive systems J. Stat. Phys. 83, 385, (1996)

Mention other references.

⊙ Systems with many degrees of freedom



Active system

Questions: ① What are average density, ~~Temp~~ Energy profile: $\langle \rho(x,t)\rangle$, $\langle E(x,t)\rangle$

② Average flow: $\langle J(x,t)\rangle$

③ Correlations: $\langle \rho(x,t)\,\rho(y,t')\rangle$; $\langle J(x,t)\,J(y,t')\rangle$

④ Stationary measure: $P(\rho(x))$, $P(J(x))$

⑤ Phase transition, spontaneous symmetry breaking, "collective behavior"

A description in Macroscopic (coarse-grain) scale

Hydrodynamics → average + fluctuation

Micro ⟶ Macro

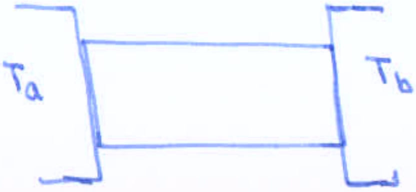
"Langevin equation for fields"

ONE-DIMENSION

Diffusive systems

"Predominant transport mechanism"

Fourier law (1822)



large L and T

$$\frac{\langle Q_T \rangle}{T} \simeq \frac{D(\bar{T})}{L} \cdot (T_a - T_b)$$

decays with system length.

Examples:

⑥ Typically where internal dynamics is coupled with bath.

① Colloids in fluids
② [illegible] conductors.
③ Interacting particles
④ Active particles.
⑤ Traffic models

Ficks law (1855)

large L and T

$$\frac{\langle Q_T \rangle}{T} = \frac{D(\rho)}{L} \cdot (\rho_a - \rho_b)$$

Remark: Anomalous transport in 1-d

Ex: Ideal gas
harmonic chain
anharmonic chain

$$\langle J \rangle \sim \frac{1}{L^{1-\alpha}} \qquad 0 < \alpha < 1$$

Ref: ① Dhar. Heat transport in low-dimensional systems. Advances in Physics 2008

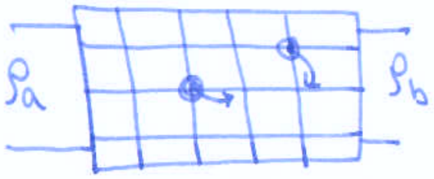
② Lepri, Livi, Politi. Thermal conduction in classical low-dimensional lattices. Physics Reports 2003.

Theoretical Models: (we shall consider)

① Interacting Langevin particles

$$\dot{x}_i(t) = F(x_i(t)) - \frac{d}{dx_i} \sum_j U(x_i - x_j) + \eta_i$$

② Particles on lattice



(a) Non-interacting particles:

(b) Simple exclusion Process:] Derrida (2007) → Non equilibrium steady states J. Stat. mech. P07023

Evans, Hanney JPA (2005) 38, R195 ← (c) Zero-range process :

(d) Ising model with spin exchange

$$E = -J \sum n_i n_j \qquad \text{exchange} \propto e^{-\beta E}$$

* Same critical behavior of liquid-gas transition in real fluid*

(e) Heat conduction: Kipnis, Marchioro, Presutti model

$$\{e_i, e_j\} \longrightarrow \{(1-\overset{\nearrow \text{random number}}{r})(e_i+e_j),\ r(e_i+e_j)\}$$

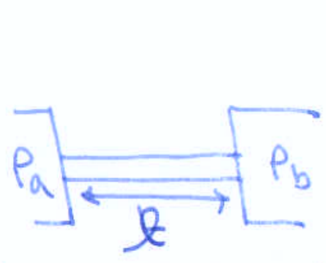
Ref: Heat flow in an exactly solvable model J. stat. Phys. 27, 65 (1982)

"Start with examples"

Transport coefficients : "One dimension"

lux t and l $\quad \dfrac{\langle Q_t\rangle}{t} \simeq \dfrac{D(\rho)}{\ell}(\rho_a-\rho_b)$ Non-equilibrium

Equilibrium $\quad \dfrac{\langle Q_t^2\rangle - \langle Q_t\rangle^2}{t} = \dfrac{\sigma(\rho)}{\ell}$



$\ell \gg \xi \gg 1$

[can verify for non-interacting particles]

Diffusivity and mobility

✱ Two are related by $\overset{\text{Kubo}}{\text{fluctuation-Response}}$ relation.

$$\boxed{\frac{\langle Q_t\rangle}{t(\rho_a-\rho_b)} \propto \frac{\langle Q^2\rangle_c}{t}}$$

local-equilibr

⊛ Idea is to write large-scale fluctuations ("slow modes")

$$J(x,T) \simeq -D(\rho(x,T))\,\frac{\partial \rho(x,T)}{\partial x} + \xi(x,T)$$

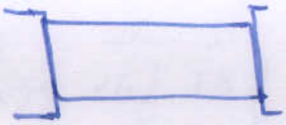
with $\langle \xi(x,T)\rangle = 0$

$$\langle \xi(x,T)\,\xi(y,T')\rangle = \sigma(\rho(x,T))\,\delta(x-x')\,\delta(T-T')$$

~~Check:~~ ① ~~$\langle J(x,T)\rangle = -\langle D(\rho(x,T))\frac{\partial\rho}{\partial x}\rangle$~~ [illegible]

In addition $\partial_t \rho = -\partial_x J$

[illegible]



Exercise: ① Check:

$$Q_{\tilde T} = \frac{1}{\ell}\int_0^{\ell} dx \int_0^{\tilde T} dT\, J(x,T)$$

SKIP

① $$\langle Q_{\tilde T}\rangle = -\frac{1}{\ell}\int_0^{\tilde T} dT \int_0^{\ell} D(\rho(x,T))\,\frac{\partial\rho}{\partial x}\,dx \simeq \frac{D(\rho(x,T))\cdot \tilde T(\rho_a-\rho_b)}{\ell}$$

② $$\langle Q_{\tilde T}^2\rangle_c = \frac{1}{\ell^2}\int_0^{\ell} dx \int_0^{\ell} dy \int_0^{\tilde T} dT \int_0^{\tilde T} dT'\, \sigma(\rho(x,T))\cdot\delta(x-y)\,\delta(T-T')$$

$$= \frac{1}{\ell^2}\int_0^{\ell} dx \int_0^{\tilde T} dT\, \sigma(\rho(x,T))$$

$$\simeq \sigma(\bar\rho)\cdot\frac{\tilde T}{\ell}$$

Argument: $\ell \gg$ correlation length for fast modes | ~~continuity~~

⊛ In addition: local equilibrium

$\Rightarrow$ $D(\rho)$ and $\sigma(\rho)$ are related.

Rescaled coordinates: coarse-graining

$$x = \frac{X}{l} \; ; \quad t = \frac{T}{l^2}$$

(Diffusive scaling

any fluctuation

- $\rho(X,T) \simeq q(x,t) + \Theta\left(\frac{1}{l}\right)$

$$\left[* \quad \frac{n_l}{l} = \frac{1}{l}\int_{-l/2}^{l/2} dX \, \rho(X,T) \right]$$

$$\simeq q(x,t) + \Theta\left(\tfrac{1}{l}\right)$$

$$\lim_{l \to \infty} \int_{-l/2}^{l/2} \frac{dX \, \rho(X,T)}{l} = q(x,t)$$

- $J(X,T) \simeq \frac{1}{l} j(x,t) + \Theta\left(\frac{1}{l^2}\right)$

$\lim \frac{1}{l}$

because current decays with system length.

Using together:

$$J(X,T) = -D(\rho(X,T))\,\partial_X \rho(X,T) + \xi(X,T)$$

$$\Rightarrow \quad j(x,t) = -D(q(x,t))\,\partial_x q(x,t) + \underbrace{l\,\xi(X,T)}_{\eta(x,t)} \Rightarrow \xi(X,T) = \frac{1}{l}\eta(x,t)$$

- $\langle \eta(x,t) \rangle = l \langle \xi(X,T) \rangle = 0$
- $\langle \eta(x,t)\eta(y,t') \rangle = l^2 \langle \xi(X,T)\xi(Y,T') \rangle$

$$= l^2 \, \sigma(\rho(X,T))\, \delta(X-Y)\, \delta(T-T')$$

$$= \frac{l^2}{l^3} \cdot \sigma(q(x,t))\, \delta(x-y)\, \delta(t-t')$$

* "correlation of fast modes which are relaxed to local equilibrium"

$$\Rightarrow \quad \boxed{\langle \eta(x,t)\eta(y,t') \rangle = \frac{1}{l}\sigma(q(x,t))\, \delta(x-x')\, \delta(t-t')}$$

⊛ [noise strength decrease $\sim \frac{1}{\sqrt{l}}$]

- Continuity: $\partial_T \rho = -\partial_X J$

$$\Rightarrow \quad \boxed{\partial_t q = -\partial_x j}$$

weak noise

Together:

$$\boxed{\partial_t q = \partial_x \left[D(q)\,\partial_x q - \eta \right] \text{ with } \langle \eta \eta \rangle = \frac{\sigma}{l} \delta(x-x')\, \delta(t-t')}$$

⊙ A phenomenological approach.

Can be rigorously proved for just few examples.

Example 1: ~~non-interacting ~~Brownian~~ particles~~

Interacting Brownian particles.

$$\boxed{\frac{dX_i(\tau)}{d\tau} = \underbrace{F(x_i) - \partial_{x_i}\sum_j V(x_i - x_j)}_{\text{we take zero.}} + \xi_i(t)}$$

Ref: D. Dean: J. Phys. A : 29, 1996, L613

$$\boxed{\frac{dx_i(\tau)}{d\tau} = \xi_i(\tau) \quad \text{with} \quad \langle \xi_i(\tau)\,\xi_j(\tau')\rangle = 2\underset{\uparrow\,\text{take}=1}{D}\,\delta_{ij}\,\delta(\tau-\tau')}$$

- Density: $\rho(x,\tau) = \sum_i \delta(x - x_i(\tau))$

$$= \sum_i \rho_i(x,\tau)$$

not smooth

- Choose a test function:

⊙ $f(x_i) = \int dx\, \rho_i(x,\tau)\, f(x)$

$$\Rightarrow \boxed{\frac{df}{d\tau} = \int dx \cdot \frac{\partial}{\partial \tau}\rho_i(x,\tau) \cdot f(x)}$$

Ito choise

⊙ $\frac{df(x_i(\tau))}{d\tau} = f'(x_i(\tau))\,\xi_i(\tau) + \frac{1}{2} f''(x_i(\tau))\cdot 2D$

$$\Rightarrow \frac{df}{d\tau} = \int dx\, \delta(x - x_i(\tau))\left[f'(x)\,\xi_i(\tau) + D\, f''(x)\right]$$

$$= \int dx\, f(x)\left[-\frac{\partial}{\partial x}\left(\delta(x - x_i(\tau))\,\xi_i(\tau)\right) + \frac{\partial^2}{\partial x^2}\delta(x - x_i(\tau))\, D\right]$$

$$\Rightarrow \frac{\partial}{\partial T} \rho_i(x,T) = \frac{\partial^2}{\partial x^2}\left[D \rho_i(x,T)\right] - \frac{\partial}{\partial x}\left[\xi_i(T)\, \rho_i(x,T)\right]$$

sum $\sum_i$

$$\Rightarrow \partial_T \rho(x,T) = \partial_x^2\left[D \rho(x,T)\right] - \partial_x \underbrace{\left[\sum_i \xi_i(T)\, \rho_i(x,T)\right]}_{\eta(x,T)}$$

$$\eta(x,T) = \sum_i \xi_i(T)\, \delta(x - x_i(T))$$

$$\langle \eta(x,T) \rangle = 0$$

$$\begin{aligned} \langle \eta(x,T)\, \eta(y,T') \rangle &= \sum_i \sum_j \langle \xi_i(T)\, \xi_j(T') \rangle\, \delta(x - x_i(T))\, \delta(y - x_i(T)) \\ &= 2D \cdot \sum_i \sum_j \delta_{ij}\, \delta(T-T')\, \delta(x - x_i(T))\, \delta(y - x_j(T')) \\ &= 2D \cdot \delta(T-T')\, \delta(x-y) \sum_i \delta(x - x_i(T)) \\ &= 2D\, \rho(x,T)\, \delta(T-T')\, \delta(x-y) \end{aligned}$$

$$\Rightarrow \boxed{\partial_T \rho(x,T)} = \partial_x^2\left[D\rho(x,T)\right] - \partial_x \eta(x,T)$$

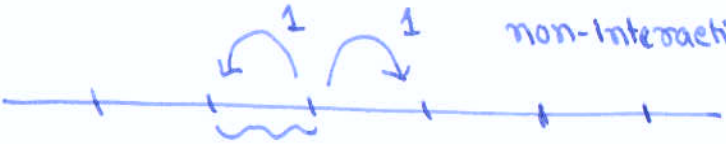
$$\boxed{= -\partial_x\left[-D\,\partial_x \rho + \eta(x,T)\right]}$$

$$\text{where } \langle \eta(x,T)\, \eta(y,T') \rangle = \underbrace{\overset{\to \text{ take } D=1}{2D\rho(x,T)}}_{\sigma(\rho)} \cdot \delta(T-T')\, \delta(x-y)$$

Remark: ① $\frac{2D}{\sigma} = \frac{1}{\rho} = +\frac{d^2}{d\rho^2}\left[+\rho \ln \rho - \rho\right]$

② can be generalized for inter-particle interactions.

⊙ **Example 2:** On lattice



non-interacting particles

⊙ Conservation of ptls:

$$n_i(T+dt) - n_i(T) = \underset{\text{net current between site } i,i+1}{J_i(T)}\, dt - J_{i+1}(T)\, dt$$

$$\Rightarrow \boxed{\frac{d\, n_i(T)}{dT} = J_i(T) - J_{i+1}(T)}$$

⊙ Average

$$\langle J_i(T)\rangle = \lim_{dt\to 0} \frac{\langle \#\ \text{ptl}\rangle \text{ flow in } dT}{dT}$$

$$= \lim_{dt\to 0} \frac{\langle n_i(T)\rangle dT - \langle n_{i+1}(T)\rangle dT}{dT}$$

$$\boxed{\langle J_i(T)\rangle = \langle n_i(T)\rangle - \langle n_{i+1}(T)\rangle}$$

⊙ Fluctuation:

$$J_i(T) = \underbrace{n_i(T) - n_{i+1}(T)}_{\text{Fick's law}} + Y_i(T)$$

One can show:

$$\langle Y_i(T)\rangle = 0$$

$$\langle Y_i(T) Y_j(T')\rangle = \left[\langle n_i(T)\rangle + \langle n_{i+1}(T)\rangle\right] \delta_{i,j}\, \delta(T-T')$$

Ref: Sadhu, Derrida
2016, Jstat mech.

⊙ Rescaling:

$$x = \frac{i}{\ell}\ ;\ t = \frac{T}{\ell^2}\ ;\quad n_i(T) \Rightarrow q(x,t)$$

$$J_i(T) \Rightarrow \frac{1}{\ell}\, j(x,t)$$

$$Y_i(T) \simeq \frac{1}{\ell}\, \eta(x,t)$$

$$\Rightarrow \boxed{j(x,t) = -\partial_x q(x,t) + \eta(x,t)}$$

Covariance:

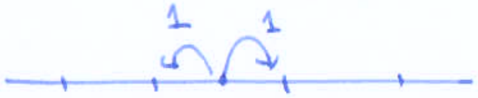
$$\boxed{\langle \eta(x,t)\,\eta(y,t')\rangle = \frac{\sigma(q)}{\ell}\cdot\delta(x-y)\,\delta(t-t')}$$

where $\sigma(q) = 2q$

* Same as Langevin equation case *

* D is absorbed in the time scale *

Similar proof for symmetric exclusion process:



- $\frac{d\tau_i(\tau)}{d\tau} = J_i(\tau) - J_{i+1}(\tau)$

- $\langle Y_i(\tau)\,Y_j(\tau)\rangle = \left[\langle \tau_i^{(\tau)}\rangle + \langle \tau_i(\tau)\rangle - 2\langle \tau_i(\tau)\,\tau_{i+1}(\tau)\rangle\right]\delta_{ij}\,\delta(\tau-\tau')$

$$\Rightarrow \boxed{\langle \eta(x,t)\,\eta(y,t')\rangle = 2\rho(x,t)\big(1-\rho(x,t)\big)\cdot\delta(x-y)\,\delta(t-t')}$$

For other systems no rigorous proof

○ Starting point:

$$\partial_t q(x,t) = -\partial_x J(x,t)$$

$$J(x,t) = -D(q)\,\partial_x q(x,t) + \eta(x,t)$$

$$\langle \eta(x,t)\,\eta(y,t')\rangle = \frac{\sigma(q)}{l}\,\delta(x-y)\,\delta(t-t')$$

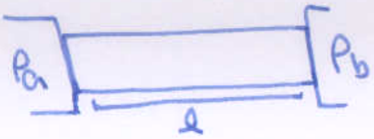
○ We want to calculate: $\langle q(x,t)\rangle, \; \langle q(x,t)q(y,t)\rangle_c \cdots$

$$P_s(q(x)) \sim e^{-l\Phi(q(x))}$$

* also applies for other fields, energy density, Angle!

⊛ In the hydrodynamic scale microscopic details are in the two transport coefficients $D(q)$ and $\sigma(q)$.

⊛ How does one calculate them?



Approach 1:

$$\frac{\langle Q_t\rangle}{t} = \frac{D(\rho)}{l}(\rho_a-\rho_b)$$ stationary state

$$\frac{\langle Q_t^2\rangle - \langle Q_t\rangle^2}{t} = \frac{\sigma(\rho)}{l}$$

Approach 2: fluctuation dissipation relation

$$\boxed{\frac{2D(\rho)}{\sigma(\rho)} = \frac{f''(\rho)}{kT}} = \frac{1}{\rho^2\chi\cdot kT} \leftarrow \text{set } k_BT=1$$

Difficulty is usually $\sigma(\rho)$ ⟵

where $f(\rho) = \frac{1}{L}\log Z_n(N,L)$

canonical free energy (Helmholtz)

Proof: later

Examples: Non interacting particles:

$$\langle J_i^{(t)}\rangle = \langle n_i^{(t)}\rangle - \langle n_{i+1}^{(t)}\rangle$$

$$\Rightarrow \sum_i \langle Q_T\rangle = \int_0^T dt \sum_i \langle J_i(t)\rangle \simeq T\cdot\left(\rho_a-\rho_b\right)$$

stationary state

$$\boxed{\langle Q_T\rangle \simeq \frac{T}{L}\left(\rho_a-\rho_b\right)} \Rightarrow \boxed{D=1}$$

- What is $f(\rho)$:

all configurations are equally probable

$$\Rightarrow Z(N,L) = \frac{L^N}{N!}$$

free energy $F = -\ln Z = -\left[N\ln L - \ln N!\right]$

$$\simeq L\left\{\rho\ln\rho - \rho\right\}$$

$$\Rightarrow \boxed{f(\rho) = \rho\ln\rho - \rho}$$

$$\Rightarrow \boxed{\sigma = \frac{2D}{f''(\rho)} = 2\rho}$$

② Symmetric exclusion process:

Ⓐ $\langle J_i\rangle = \langle n_i(1-n_{i+1})\rangle - \langle(1-n_i)n_{i+1}\rangle$

$= \langle n_i\rangle - \langle n_{i+1}\rangle$

$\longrightarrow \boxed{D=1}$

Ⓑ $Z(N,L) = \binom{L}{N} \Rightarrow f(\rho) = \rho\ln\rho + (1-\rho)\ln(1-\rho)$

(c) $\boxed{\sigma = 2\rho(1-\rho)}$

Ex3: Excereise:

next nearest neighbor exclusion.

$$\langle J_i \rangle = \langle n_i(1-n_{i+1})(1-n_{i+2})\rangle - \langle n_{i+1}(1-n_i)(1-n_{i-1})\rangle$$

$$\boxed{D(\rho) = \frac{1}{(1-\rho)^2}} \qquad f(\rho) = \rho \ln \rho + (1-2\rho)\ln(1-2\rho) - (1-\rho)\ln(1-\rho)$$

$$\sigma(\rho) = \frac{2\rho(1-2\rho)}{1-\rho}$$

Ex4: zero range process

Ex5: Boundary driven Ising model

Ex6: KMP model

$$D(\rho) = 1 \qquad \sigma(\rho) = 2\rho^2$$

Variational formulation and proof of $\frac{2D}{\sigma} = f''(\rho)$

Path measure:

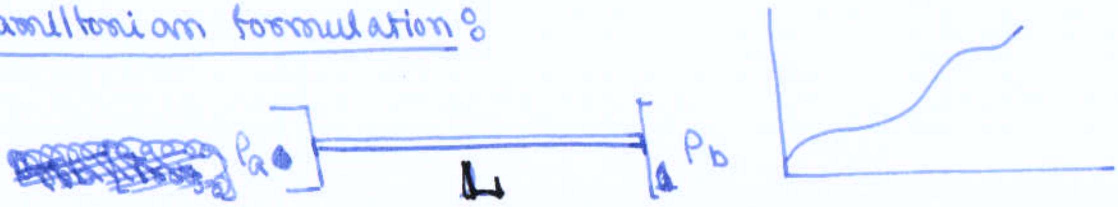
$$\text{Prob}(\text{Traj}) \sim e^{-L\int dt \int dx \frac{[\dot{J} + D(q)\partial_x q]^2}{2\sigma(q)}}$$

$\partial_t \rho = -\partial_x J$

$J = -D\partial_x \rho + \eta$

Remark: Ito-Stratonovich choise do not matter

Hamiltonian formulation:



$$P\left[q(x,T)=\gamma(x),\ q(x,0)=\rho(x)\right]$$

$$= \int_{\rho(x)}^{\gamma(x)} \mathcal{D}[q,J]\, e^{-L\int_0^T dt \int^1 dx \frac{[J + D(q)\partial_x q]^2}{2\sigma(q)}}\ \delta(\partial_t q + \partial_x J)$$

$$= \int \mathcal{D}[q,J,p]\, e^{-L\int_0^T dt \int_0^1 dx \frac{(J + D(q)\partial_x q)^2}{2\sigma(q)} - L\int_0^T dt \int_0^1 dx\, p\,(\partial_t q + \partial_x J)}$$

$$= \int \mathcal{D}[q,J,p]\, e^{-L\int dt \int dx \left\{ p\dot{q} - J\cdot\partial_x p + \frac{(J + D\partial_x q)^2}{2\sigma(q)} \right\}}$$

~~assuming~~ using $p(0)=0$, $p(1)=0$ } no condition at the boundary

$$= \int \mathcal{D}[q,p]\, e^{-L\int dt\int dx\, p\dot{q}} \int \mathcal{D}[J]\cdot e^{-L\int dt \int dx \left\{ \frac{[J + (Dq' - \sigma p')]^2}{2\sigma} \right\}}$$

$$e^{-L\int dt \int dx \frac{D^2(\partial_x q)^2 - (Dq' - \sigma p')^2}{2\sigma}}$$

$$\simeq \int \mathcal{D}[q,p]\, e^{-L\int dt \int dx \left\{ p\dot{q} - \frac{\sigma^2 p'^2 - 2\sigma D p' q'}{2\sigma} \right\}}$$

$$\simeq \int \mathcal{D}[p,q]\, e^{-L\int dt \int dx \cdot \left\{ p\dot{q} - \underbrace{\left[\frac{\sigma}{2}(\partial_x p)^2 - D(q)\cdot \partial_x q \cdot \partial_x p \right]}_{H[p,q]} \right\}}$$

$$\text{Prob}[\gamma(x), \rho_0(x)] \simeq \int \mathcal{D}[p,q]\, e^{-L\, S[p,q]}$$

with $S[p,q] = \int dt \left\{ \int dx\, p\dot{q} - H[p,q] \right\}$

where $\boxed{H[p,q] = \frac{\sigma(q)}{2}(\partial_x p)^2 - D(q)\cdot(\partial_x q)\cdot(\partial_x p)}$

Equilibrium:

Stat mech

$$P[q(x) = \gamma(x)] \asymp e^{-L\,\phi[\gamma(x)]}$$

where $\phi[\gamma] = \int_0^1 dx \left\{ f(\gamma) - f(\rho) - f'(\rho)[\gamma - \rho] \right\}$

Derivation using the Action formulation:

Similar approach as used for Langevin equation

$$P[\gamma(x)] = P[q(x,0) = \gamma(x);\, q(x,-\infty) = \rho]$$

$$= \int_{q(x,-\infty)=\rho}^{q(x,0)=\gamma(x)} \mathcal{D}[p,q]\, e^{-L\, S[p,q]}$$

$$\Rightarrow \boxed{\phi[\gamma(x)] = \min S[p,q]}$$

Variational calculus:

$$S[p,q] = \int_{-\infty}^{0} dt \left\{ \int_0^1 dx\, p\dot{q} - H[p,q] \right\}$$

$$\hookrightarrow H = \int dx \{ \qquad \}$$

SKIP

$$\Rightarrow \delta S[p,q] = \int_{-\infty}^{0} dt \int_0^1 dx \left\{ \left(\dot{q} - \frac{\delta H}{\delta p}\right) \delta p(x,t) - \left(\dot{p} + \frac{\delta H}{\delta q}\right) \delta q \right\}$$

$$+ \int_0^1 dx\, p(x,T)\, \underbrace{\delta q(x,T)}_{0}$$

$$- \int_0^1 dx\, p(x,0)\, \underbrace{\delta q(x,0)}_{0}$$

Least Action paths:

$$\dot{q} = \frac{\delta H}{\delta p} = -\partial_x \left[\frac{\sigma(q)}{\cancel{2}} \cdot \cancel{2} \cdot (\partial_x p) - D(q)\, \partial_x q \right]$$

$$\Rightarrow \boxed{\partial_t q - \partial_x (D(q)\, \partial_x q) = -\partial_x (\sigma(q)\, \partial_x p)}$$

$$\dot{p} = -\frac{\delta H}{\delta q} = -\left\{ \frac{\sigma'(q)}{2} (\partial_x p)^2 - D'(q) \cdot \partial_x q \cdot \partial_x p + \partial_x (D(q)\, \partial_x p) \right\}$$

$$\Rightarrow \partial_t p = -\frac{\sigma'(q)}{2} (\partial_x p)^2 + \partial_x D(q) \cdot \partial_x p - \partial_x (D(q)\, \partial_x p)$$

$$\Rightarrow \boxed{\partial_t p + D(q)\, \partial_x^2 p = -\frac{\sigma'(q)}{2} (\partial_x p)^2}$$

With boundary conditions: $\boxed{q(x,0) = \rho(x) \text{ and } q(x,-\infty) = \rho}$

In addition:
$$\boxed{\begin{array}{l} q(0,t) = \rho \text{ and } q(1,t) = \rho \\ p(0,t) = 0 \text{ and } p(1,t) = 0 \end{array}}$$

Solution:

$$P(x,t) = \int_{\rho}^{q(x,t)} ds\, \frac{2D(s)}{\sigma(s)} \quad \text{(ansatz)}$$

(* why lower limit? because we want $P=0$ at $x=0,1$

then

→ Check one has to solve only one equation.

[$\cdot\ \partial_x P = \frac{2D}{\sigma}\cdot\partial_x q \Rightarrow \boxed{\sigma\,\partial_x P = 2D\,\partial_x q}$

$$\Rightarrow \partial_t q = \partial_x(D\partial_x q) - \underbrace{\partial_x(\sigma\partial_x P)}_{2\partial_x(D\partial_x q)}$$

first

$$\boxed{\partial_t q = -\partial_x(D\partial_x q)} \quad \boxed{\text{time reversed equation.}}$$

$$\Rightarrow \partial_t P = \frac{2D}{\sigma}\cdot\partial_t q = -\frac{2D}{\sigma}\,\partial_x[D\partial_x q]$$

$$= -\frac{\cancel{2}D}{\sigma}\cdot\frac{1}{\cancel{2}}\cdot\partial_x(\sigma\partial_x P)$$

$$= -D\partial_{xx}P - \frac{D}{\sigma}\cdot\sigma'\cdot\partial_x q\cdot\partial_x P$$

$$= -D\partial_{xx}P - \frac{\sigma'}{\cancel{\sigma}}\cdot\left[\frac{\cancel{\sigma}}{2}\cdot\partial_x P\right]\cdot\partial_x P$$

$$\Rightarrow \boxed{\partial_t P = -D\partial_{xx}P - \frac{\sigma'}{2}(\partial_x P)^2}$$

automatically satisfied.]

⊙ **Boundary condition:**

$$\partial_t q = -\partial_x(D\partial_x q)$$

with $q(x,-\infty) = \rho_0$ and $q(x,0) = \gamma(x)$

- **Check:** $H[P,q] = \int\left[\frac{\sigma(q)}{2}(\partial_x P)^2 - \underbrace{D(q)\cdot\partial_x q\cdot\partial_x P}_{\hookrightarrow\ \sigma/2\cdot\partial_x P}\right]dx$

$= 0$

Then

$$\Phi[\gamma(x)] = \int_{-\infty}^{0} dt \int_{0}^{1} dx \; p \, \dot{q}$$

lets write

$$\frac{2D(s)}{\sigma(s)} = f_1''(s) \Rightarrow p(x,t) = \int_{\rho}^{q(x,t)} ds \; f_1''(s)$$

$$= f_1'(q(x,t)) - f_1'(\rho)$$

Substitute

$$\Phi[\gamma(x)] = \int_{0}^{1} dx \int_{-\infty}^{0} dt \; [\, f_1'(q) - f_1'(\rho)\,]\dot{q}$$

$$= \int_{0}^{1} dx \int_{-\infty}^{0} dt \left[\frac{d}{dt} f_1(q(x,t)) - f_1'(\rho)\cdot\dot{q} \right]$$

$$= \int_{0}^{1} dx \cdot \left[f_1(\gamma(x)) - f_1(\rho) - f_1'(\rho)(\gamma - \rho) \right]$$

Compare with result from ~~Stat~~ equilibrium statistical mechanics

$\Rightarrow$ $f(s)$ = equilibrium free energy.

$$\Rightarrow \boxed{\frac{2D(s)}{\sigma(s)} = f''(s)}$$

Q: Why does this break down for non-equilibrium!

$\rho \leftrightarrow \rho(x)$ solution does not work

⊛ Only case it works ≡ Non-interacting particles.

Non-equilibrium, Non-interacting Particles

$$\boxed{\begin{aligned} D(q) &= 1 \\ \sigma(q) &= 2q \end{aligned}}$$

$$\bar{\rho}(x) = \rho_a(1-x) + \rho_b x$$

(Solve: $\frac{\partial \rho}{\partial t} = \partial_{xx} \rho$

with $\rho(0,t) = \rho_a$; $\rho(1,t) = \rho_b$)

To show:

$$P(\gamma(x)) \approx e^{-L\, F[\gamma(x)]}$$

$$F[\gamma(x)] = \int_0^1 dx \left\{ \gamma(x) \ln \frac{\gamma(x)}{\rho(x)} - \gamma(x) + \rho(x) \right\}$$

Solution of the Hamilton's equation

$$[redacted]\ \partial_t p + \partial_{xx} p = -(\partial_x p)^2$$

$$\partial_t q - \partial_{xx} q = -2\partial_x (q\, \partial_x p)$$

Solution:

$$p(x,t) = \int_{\rho(x)}^{q(x,t)} ds \frac{1}{s} = \ln q(x,t) - \ln \rho(x)$$

[redacted]

[leave as an exercise to show

$$\partial_t q = -\partial_{xx} q + 2\partial_x \left[q \cdot \frac{\partial_x \rho}{\rho} \right]$$

$\partial_t p$ eqⁿ is automatically satisfied!

Check: $\partial_x p = \frac{\partial_x q}{q} - \frac{\partial_x \rho}{\rho} \Rightarrow q\, \partial_x p = \partial_x q - q \cdot \frac{\partial_x \rho}{\rho}$

- $$\partial_t q = \partial_{xx} q - 2\,\partial_x(q\,\partial_x p)$$

$$= \partial_{xx} q - 2\,\partial_{xx} q + 2\,\partial_x\left(q\,\frac{\partial_x \rho}{\rho}\right)$$

$$\Rightarrow \boxed{\partial_t q + \partial_{xx} q = 2\,\partial_x\left(q\,\frac{\partial_x \rho}{\rho}\right)}$$

- $$\partial_t p = \frac{\partial_t q}{q} = \frac{1}{q}\left\{-\partial_{xx} q + 2\,\partial_x\left(q\,\frac{\partial_x \rho}{\rho}\right)\right\}$$

use $q\partial_x p = \partial_x q - q\frac{\partial_x \rho}{\rho}$

$$= \frac{1}{q}\left\{-\partial_x(q\,\partial_x p) - \partial_x\left(q\,\frac{\partial_x \rho}{\rho}\right) + 2\,\partial_x\left(q\,\frac{\partial_x \rho}{\rho}\right)\right\}$$

$$= -\partial_{xx} p - \frac{1}{q}\cdot(\partial_x q\cdot)(\partial_x p) + \frac{1}{q}\partial_x\left(q\,\frac{\partial_x \rho}{\rho}\right)$$

$$\Rightarrow \partial_t p + \partial_{xx} p = -\frac{1}{q}(\partial_x q)\cdot(\partial_x p) + \frac{\partial_x q}{q}\cdot\frac{\partial_x \rho}{\rho} - \frac{\cancel{q}}{\cancel{q}}\cdot\frac{(\partial_x \rho)^2}{\rho^2} + \frac{\cancel{q}}{\cancel{q}}\cdot\frac{\overset{\overset{0}{\uparrow}}{\boxed{\partial_x^2 \rho}}}{\rho}$$

$$= -\frac{1}{q}\partial_x p\cdot\left(q\,\partial_x p + q\,\frac{\partial_x \rho}{\rho}\right) + \frac{\partial_x q\cdot\partial_x \rho}{q\,\rho} - \cdot\frac{(\partial_x \rho)^2}{\rho^2}$$

$$= -(\partial_x p)^2 - (\partial_x p)\cdot\underbrace{\frac{\partial_x \rho}{\rho} + \frac{\partial_x q}{q}\cdot\frac{\partial_x \rho}{\rho} - \cdot\frac{(\partial_x \rho)^2}{\rho^2}}$$

$$\Rightarrow -\frac{1}{q}\cdot\partial_x q\cdot\frac{\partial_x \rho}{\rho} + \left(\frac{\partial_x \rho}{\rho}\right)^2 + \frac{\partial_x q}{q}\cdot\frac{\partial_x \rho}{\rho} - \frac{(\partial_x \rho)^2}{\rho^2} = 0$$

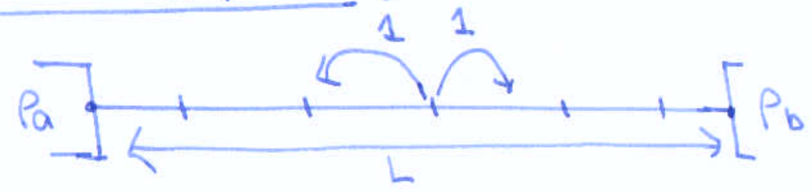
Then follow the earlier procedure.

$$\Phi[\gamma(x)] = \int dx\left\{f(\gamma) - f(\rho) - f'(\rho)\cdot(\gamma-\rho)\right\} \text{ with } f(\gamma) = \gamma\ln\gamma - \gamma$$

$$= \int dx\left\{\gamma\ln\frac{\gamma}{\rho} - \gamma + \rho\right\}$$

⊛ local functional out-side equilibrium

⊙ **Symmetric exclusion process:**



$$D(\rho)=1 \;;\quad \sigma(\rho)=2\rho(1-\rho)$$

Average profile: $\rho(x) = \rho_a(1-x) + x\rho_b$

Profile at stationary state: $P\big(\gamma(x)\big) \asymp e^{-L\,\Phi[\gamma(x)]}$

Equilibrium: $\rho_a=\rho_b=\rho$

$$\boxed{\Phi[\gamma(x)] = \int_0^1 dx \left\{ \gamma(x)\ln\frac{\gamma(x)}{\rho} + (1-\gamma(x))\ln\frac{1-\gamma(x)}{1-\rho} \right\}}$$

"local functional"

$f(\rho)=\rho\ln\rho+(1-\rho)\ln(1-\rho)$

Non-equilibrium: $\rho_a \neq \rho_b$

$$\boxed{\Phi[\gamma(x)] = \min_{F(x)} \int_0^1 dx \left\{ \gamma(x)\cdot\ln\frac{\gamma(x)}{F(x)} + (1-\gamma(x))\ln\frac{1-\gamma(x)}{1-F(x)} + \ln\frac{\partial_x F(x)}{\rho_b-\rho_a} \right\}}$$

where $F(0)=\rho_a$ and $F(1)=\rho_b$

~~Comment 1: $\rho_a-\rho_b\to 0$: $F(x)=\rho$ ⇒ one gets back equilibrium.~~

Comment 2:

$$\boxed{F(x) + \frac{F(x)(1-F(x))}{(\partial_x F)^2}\cdot(\partial_x^2 F) = \gamma(x)}$$

Comment 2: For small $\rho_a-\rho_b$ one solves $F(x)$ perturbatively and get

$$F(x) \simeq \rho(x) - \frac{(\rho_a-\rho_b)^2}{\rho_a(1-\rho_a} \left[(1-x)\int_0^x dy \left(r(y)-\rho(y)\right) + x\int_x^1 dy\,(1-y)\left(r(y)-\rho(y)\right) \right] + \dots$$

and one gets

$$\Rightarrow \Phi[r(x)] = \cancel{\Phi_{eq}[r(x)]} \int_0^1 dx \left\{ r(x)\ln\frac{r(x)}{\rho(x)} + (1-r(x))\ln\frac{1-r(x)}{1-\rho(x)} \right\}$$

$$+ \frac{(\rho_a-\rho_b)^2}{[\rho_a(1-\rho_a)]^2} \int_0^1 dx \int_0^1 dy \left[x(1-y)\left(r(x)-\rho(x)\right)\left(r(y)-\rho(y)\right) \right]$$

$$+ (\rho_a-\rho_b)^3 \ast \dots$$

⊛ Non-local function ⇒ correlations are long-ranged.

How to solve this

⊛ Microscopic solution: Matrix product Ansatz

① Derrida, Lebowitz, Speer: PRL. 87, 150601 (2001)

② " " " : JSP 107, 599 (2002)

⊛ Using variational formulation

① Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim
PRL 87, 040601 (2001)
JSP 107, 635 (2002)

② Tailleur, Kurchan, Lecomte PRL 99, 150602 (2007)
JPA 41, 505001 (2008)