Ising model in 2d

Tuesday, June 1, 2021 3:41 PM

Ref: Ch 5 of the book by Mussardo.

Exact solution for nearest neighbor Ising model.

Ferromagnetic Ising model on square lattice with periodic boundary

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \qquad \text{with} \qquad \sigma_i = \pm 1$$

Before we present the derivation, let's look the expression.

In the thermodynamic limit, the free energy density

$$f = -\log 2 + \log(1-v^2) - \frac{1}{8\pi^2} \int_0^{2\pi} d\omega_1 \int_0^{2\pi} d\omega_2 \log\left[(1+v^2)^2 - 2v(1-v^2)(\cos\omega_1 + \cos\omega_2)\right]$$

$v = \tanh J$ [ $\beta$ is implicit ]

Singular point: ① at $v = 1 \Rightarrow J \to \infty$ (zero temperature)

② at $v_c$ where the term vanish.

$$v_c = \sqrt{2} - 1 \Rightarrow \boxed{J_c = \text{arctanh}(\sqrt{2}-1) = 0.44068}$$

This is same as $\boxed{\sinh 2J = 1}$

Near critical point $(T = T_c)$

$$f = A - B(T-T_c)^2 \log|T-T_c| + \ldots$$

a nonsingular part

has logarithmic nonanalyticity.

Specific heat

$$C_v = f''(T) \sim B \log|T-T_c|$$

not a power-law divergence

$\Rightarrow \alpha = 0$

2d Ising model critical exponents (Exact results)

$$\boxed{\alpha = 0,\ \beta = \frac{1}{8},\ \gamma = \frac{7}{4},\ \delta = 15,\ \eta = \frac{1}{4}}$$

Exact solution: ① Using graphical expansion method.

$$Z_N = 2^N (\cosh J)^{2N} \sum_{l=4,6,8} N_l \cdot (\tanh J)^l$$

$$= 2^N \frac{1}{\ldots} \sum \quad N_l \cdot v^l$$

sum over all closed loops.

$$= 2^N \frac{1}{(1-v^2)^N} \sum_{l=4,6,8} N_l \cdot v^l \qquad \text{sum over all closed loops.}$$

Kac and Ward, and later by Vdovichenko showed how to count all such graphs be deforming the closed loops and mapping to a Random walk problem.

## ② a solution by mapping to dimer filling problem. (Relates to pfaffian of a matrix)

First, dimer covering of square lattice :

$$A = \begin{bmatrix} 0 & a_{1,2} & a_{1,3} & a_{14} & \cdots & a_{1,2N} \\ -a_{12} & 0 & & & & \\ \vdots & & 0 & & & \\ -a_{1,2N} & & & & & 0 \end{bmatrix}_{2N\times 2N}$$

An $2N\times 2N$ antisymmetric matrix.

$$\det(A) = \sum_{\sigma} \text{Sgn}(\sigma)\, a_{1,\sigma_1} \cdots a_{2N,\sigma_{2N}}$$

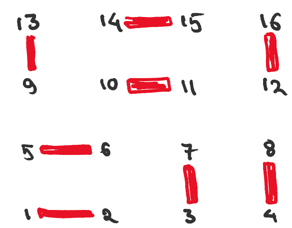
$$\text{Pf}(A) = \sum_{\sigma}{}' \text{Sgn}(\sigma)\, a_{\sigma_1\sigma_2}\, a_{\sigma_3\sigma_4} \cdots a_{\sigma_{2N-1},\sigma_{2N}}$$

[ Standard form (see Wiki) can be obtained using antisymm of A ]

The summation is over permutations with $\sigma_{2K-1} < \sigma_{2K}$ and $\sigma_{2K-1} < \sigma_{2K+1}$

Explicitly $\sigma_1 < \sigma_2$ and $\sigma_1 < \sigma_3$
$\sigma_3 < \sigma_4$ and $\sigma_3 < \sigma_5$ and so on.... $\{\cdots(\sigma_{2K-1},\sigma_{2K}),(\sigma_{2K+1},\sigma_{2K+2})\cdots\}$

Relation to dimer coverings of a square lattice such that every node has exactly one dimer associated. Question we ask is how many ($\Omega$) configurations possible for dimer arrangements?

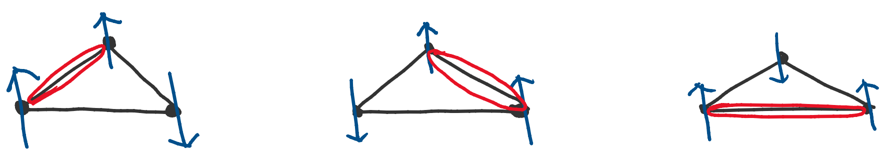



A 4x4 square lattice

Fact : for a $2L\times 2L = 2N$ sites there are N dimers.

For a $8\times 8$ : $\Omega = 2^4 (901)^2$ [Michael Fisher]

Why relevant : one example is antiferromagnetic Ising on a triangular lattice



Frustrated magnets have ground state with macroscopic degeneracy.

Frustrated magnets have ground state with macroscopic degeneracy.
For more, talk to Kedar Damle.

<u>connection to pfaffian</u>: for a dimer config associate a pair of numbers

$$(1,2),\ (3,7),\ (4,8),\ (5,6),\ (9,13),\ (10,11),\ (12,16),\ (14,15)$$

Follows the same constraints as in pfaffian.
This simply comes because dimers connect only nearest neighbours.

This means,
if we construct an antisymmetric matrix $A$ with

$$(\text{for } i<j)\quad a_{ij} = \begin{cases} z & \text{if } i,j \text{ are nearest neighbour} \\ 0 & \text{other wise} \end{cases}$$

$$\begin{array}{c c} & 1\ 2\ 3 \cdots\ \cdots 16 \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ 16 \end{matrix} & \begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

then the generating function

$$G(z) = \sum_{\text{all config}} z^N = \sum_{\text{all config}} z \cdots\cdots z$$

$$= \sum_{\text{all config}} a_{\sigma_1\sigma_2}\, a_{\sigma_2\sigma_3} \cdots a_{\sigma_{2N-1},\sigma_{2N}}$$

maintains the pfaffian constraint.

Then, it is almost like a pfaffian except the $\text{Sgn}(\sigma)$ term.

This can be done by adding a phase factor for matrix elements

[due to P.W. Kastelyn for any planar lattice]

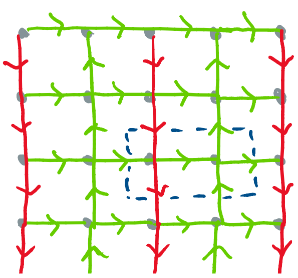
$$(\text{for } i<j)\quad a_{ij} = \begin{cases} z & \text{if } i\,j \text{ are horizontal neighbour,} \\ (-1)^i z & \text{if } i\,j \text{ are vertical neighbor,} \\ 0 & \text{elsewhere.} \end{cases}$$

This gives, the dimer generating function

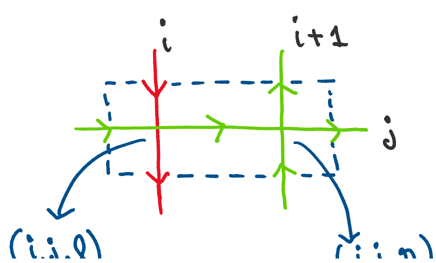
$$\boxed{G(z) = Pf(A)}$$

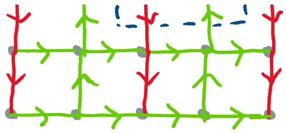
## <u>How do we calculate the pf(A)?</u>

<u>First think with geometry</u>: The sign of $a_{ij}$ can be thought interms of arrows

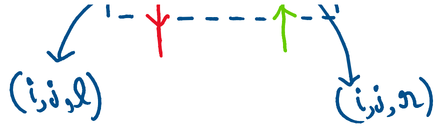


For this directed graph the unit cell is a rectangular box indicated by blue dotted box.





box.




Using this observation one can write $A$ in a $2\times 2$ block form which are cyclic matrices. Then $\det A$ is the product of det of these blocks, which allows us to calculate the $\det A$.

What we get

$$\det A = \prod_{K=0}^{2L} \det B_K \qquad B_K = \begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix}_{2\times 2}$$

Taking log and taking thermodynamic limit we set

$$\frac{1}{4L^2} \log \det A = \frac{1}{(2\pi)^2} \int_0^{2\pi} dq \, \log\left[4z^2(1-\cos q)\right]$$

$\Rightarrow$ then using $\boxed{Pf(A) = \sqrt{\det A}}$ $\Rightarrow$ $\log Pf(A) = \frac{1}{2} \log \det A$

gives the full generating function for dimer filling.

**A corollary:** number of possible dimer coverings of a square lattice of $2L\times 2L$ sites,

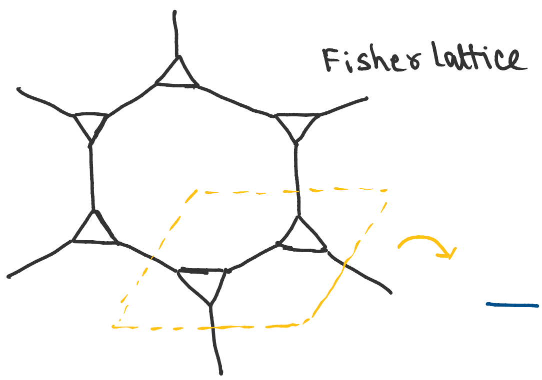
$$\Omega \sim e^{\frac{4L^2}{\pi}\cdot C}$$

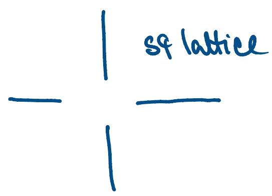
catalan numbers

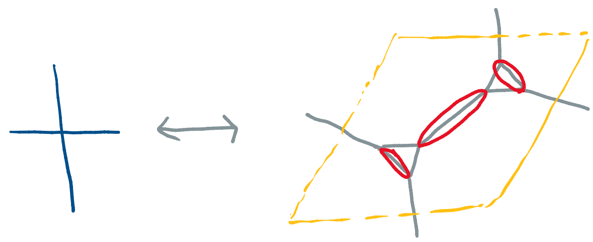
$$C = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots = 0.9159\ldots$$

**Relation to Ising model:**

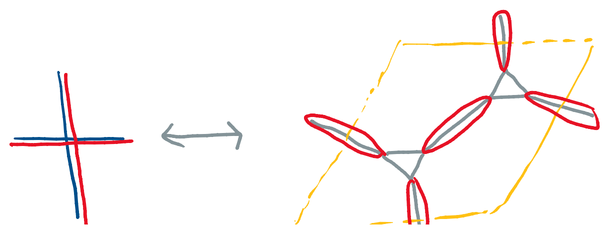
For each closed loops on square lattice (Ising graph expansion) there is a dimer config on Fisher lattice.



By this duality, Ising partion function on sq lattice can be linked to dimer generating function on Fisher lattice



can be linked to dimer generating function on Fisher lattice

$$\frac{Z_N}{(2\cosh^2 J)^N} = \sum_{\text{dim}} z^n$$

Ref: See ch 5.2.2 of the book of Mussardo.