Pattern formation in sandpile models

What are sandpile models?

Ref: (1) James Propp and Lionel Levin, in Notices of American mathematical Society, Vol 57, Page 976, 2010.

(2) Deepak Dhar, in Physica A, vol 369, page 29, 2006.

(3) Frank Redig, "Mathematical aspects of abelian sandpile model" in lecture notes of Les Houches summer school, 2005.

Bak-Tang-Wiesenfeld (BTW) model' 1987.

Abelian sandpile model by Deepak Dhar' 1990.

Illustration on a finite square grid with open boundary condition.

Configurations: Each cell can have positive integer numbers (number of sand grains). This is called height variable $z_i = 0, 1, 2, \cdots$.

stable config

2	3	2
3	3	0
1	2	3

Threshold height: A stable cell can accomodate atmost $z_c - 1$ number of grains. Means if $z_i < z_c$ then the cell is stable, and if $z \geq z_c$ it is unstable. The threshold height $z_c = 4$ for square grid.

Unstable config

2	3	2
3	4	0
1	2	3

Dynamics: If a cell is unstable, it gives away $z_c = 4$ particles to its nearest neighbor cells, one to each cell. When this happens we say that the site has toppled.

a v a l a n c h.

2	4	2
4	0	1
1	3	3

Avalance: Unstable configurations topple untill the grid reaches a stable configuration. This happens because (one reason) becaus grains once moved outside the grid by toppling at the boundary are lost.

4	0	3
0	2	1
2	3	3

A sequence of toppling untill a stable config is reached is called an avalanch.

0	1	3
1	2	1
2	3	3

In orignal BTW, unstable sites are toppled in parallel.

Slowly driven: In a stable configuration, grains are dropped at random sites untill the grid becomes unstable and an avalanch starts.

Once avalanch stops, the process is repeated.

Adding grains is called driving, and because grains are added only between avalances, the driving is slow.

Observation: We consider the stable configurations between successive avalanches.

Driving + avalanch (relaxation step) takes from one stable configuration to another stable configuration.
Avalanch step is deterministic. Randomness comes from where grains are added and in the initial configuration.

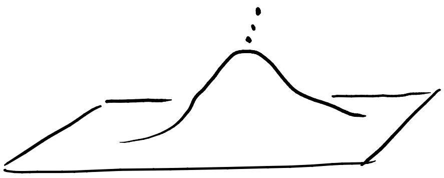
Repeating the process "many times" brings the system to a stationary state where probability of height config is time independant.

The stationary state is critical:

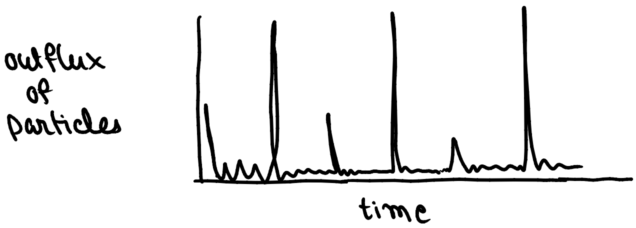
(1) height-height correlation has power-law tail.

(2) Avalanch size (number of sites toppled, duration, etc) distribution has power-law tail.

Draw analogy with a real sandpile:



$F(s) = \int_s dx\, P(x) \equiv$ cumulative probability of an avalanch of size s or larger.



burst-like relaxation.

The system under its own dynamics, without fine tuning of parameters by an external agent, self-organizes to a critical state. BTW sandpile model illustrates self organized criticality. [Per Bak]

Questions: Statistical properties in stationary state, characteristics of dynamics.

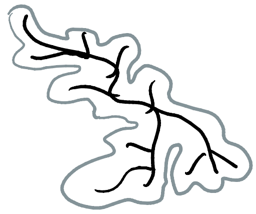
Natural examples of self organized criticality:

(1) height distribution in a mountain range.

$$\langle \Delta h(r)\, \Delta h(0) \rangle \sim r^{\delta}$$

(2) Gutenberg-Richter law in earthquake E^{-2}

(3) Hack's law in River network.



Ref: P. Bak, "how nature works", Book.

Catchment area of a river grows as powerlaw with length of the river. $A \sim \ell^{1.6}$

Abelian sandpile Model (ASM): Dhar'

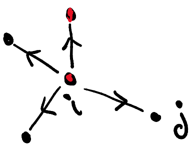
- Sandpile model on any general graph with N-nodes.
- A configuration $\quad \mathcal{C} = \{ z_i \geq 0 \}$
- Toppling rule specified by toppling matrix $\Delta_{N\times N}$, such that toppling of site i changes height

$$z_j \to z_j - \Delta_{ij}$$

for square grid

$$\Delta_{ij} = \begin{cases} 4 & \text{for } i=j \\ -1 & \text{for } i,j \text{ nearest neighbor} \\ 0 & \text{otherwise} \end{cases}$$

- Toppling matrix



(a) $\Delta_{ii} = z_c^{(i)}$ coordination number of ith node.

(b) $\Delta_{ij} \leq 0$ for all $j \neq i$ (required for Abelian property, see later)

(c) $\sum_j \Delta_{ij} \geq 0$ for all i (grains are not generated, but they can get lost at the boundary)

(d) Each site is connected by successive toppling to atleast one Sink site (Boundary)

Properties: (1) Space Ω of stable configurations $\mathcal{C}_s$ made of z_c^N configs.

(2) By avalanch system remains in Ω. No negative heights are generated, and stability is reached in finite steps (due to property (d) of Δ)

(3) Order of topplings do not matter. [Deepak Dhar 1990]

1 dimension

|0|2|1|3|0|

|1|0|2|3|0| $\quad$ |0|2|2|1|1|

|1|0|3|1|1|

① Toppling at i and j site changes height at $k \neq i \neq j$ by an amount $-\Delta_{ik} - \Delta_{jk}$

this is indep of order of toppling.

② Toppling at i changes height at j by $-\Delta_{ij} \geq 0$ (condition b) therefore, it can not make the jth site stable. Same applies for ith site affected by toppling at jth site.

$$\Rightarrow z_j \to z_j - \Delta_{jj} - \Delta_{ij} \quad \text{same irrespective of order}$$

$\Rightarrow z_j \to z_j - \Delta_{jj} - \Delta_{ij}$ same irrespective of order of toppling.

(4) Abelian property: $a_i \equiv$ addition operator corresponding to addition of sandgrain at site i.

$\boxed{a_i\, e_s = e_s'}$ $\equiv$ stable configuration reached by adding a grain at site i in config e_s and then relaxing.

Deepak Dhar noticed an Abelian property

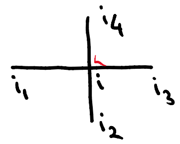
$\boxed{[a_i, a_j] = 0} \quad \Rightarrow \quad a_i a_j e_s = a_j a_i e_s$

This can be seen by noting that addition operation and toppling at unstable site commute.

[See Redig's lecture note for a more mathematical proof.]

(5) Operator algebra:

$$a_i^4 = a_{i_1} a_{i_2} a_{i_3} a_{i_4}$$ for square grid.



In general $\boxed{a_i^{\Delta_{ii}} = \prod_{j \neq i} a_j^{-\Delta_{ij}}}$

(5) "Garden of Eden" configurations:

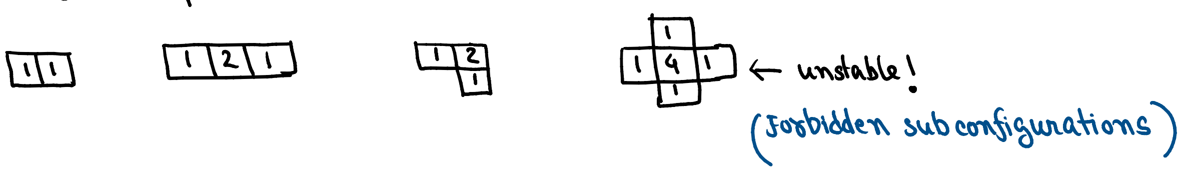
Stable configurations which once exited can never be reached again by addition operation (drive + relaxation)

Example:

2	3	2
3	0	0
1	2	3

Also referred as transient configurations.

Others examples



(6) Then the stable configuration space is decomposed into a transient space $\mathcal{T}$ and recurrent space $\mathcal{R}$.

(7) Space $\mathcal{R}$ is not-empty. It has atleast the minimally stable configuration

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$$z_i = z_c - 1 \quad \text{for all } i.$$

(8) On this recurrent space a_i^{-1} is unique, ie

$$a_i\left(a_i^{-1} C_s\right) = C_s \qquad \forall\ C_s \in \mathcal{R}.$$

simply because, $\|R\|$ is finite, and therefore

$$a_i^{n} C_s = C_s \quad \text{for a finite positive integer } n.$$

$$\Rightarrow a_i\left(a_i^{n-1}\right) C_s = C_s \Rightarrow a_i^{-1} = a_i^{n-1} \quad \text{exist.}$$

Moreover,

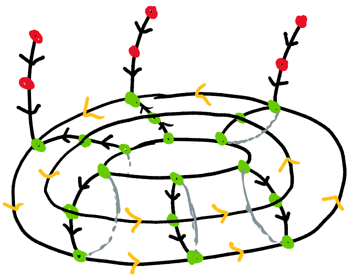
$$a_i^{n} \underbrace{a_j C_s}_{C_s'} = a_j a_i^{n} C_s = \underbrace{a_j C_s}_{C_s'}$$

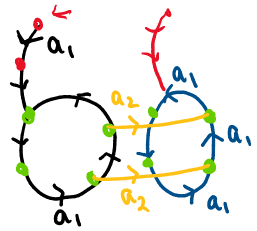
$$\Rightarrow a_i^{n} C_s' = C_s' \qquad \text{same period } n.$$

Extending this means, period n is same for all $C_s \in R$.

a_i^{-1} unique.

(9)





Repeated action of a_1 leads to disconected limit cycles which are then joined by other a_js. All cycles have same period.

Set of all recurrent configurations forms a multidimensional torus generated by operators a_i. Any recurrent config is reachable from another as long as all a_i are acted with non-zero probability. One way to see this is by noting that minimally stable configuration is recurrent and can be reached from any recurrent configuration. Reverse is also true because a^{-1} exists.

Consequence of these properties:

(1) Symmetry of the torus and that a_i are randomly acted (random addition of particle) gives that all configurations are equally probable in the stationary state.

(2) Volume of R

$$\boxed{\|R\| = \text{Det}\,\Delta}$$

② Volume of R

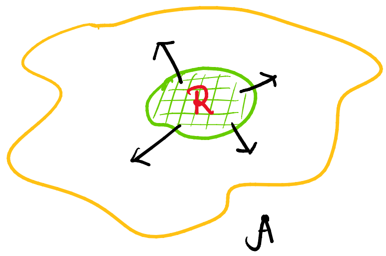
$$\boxed{\|R\| = \mathrm{Det}\,\Delta}$$

For a square grid of N sites, $\dfrac{\|R\|}{\|\Omega\|} = \left(\dfrac{3.2102\ldots}{4}\right)^N$ exponentially small subset of recurrent configurations.

These are attractor of the dynamics.

How do we calculate $\|R\|$? Deepak Dhar' 1990

R is recurrent space



$\mathcal{A}$

Space $\mathcal{A}$ includes additional configurations (stable + unstable) that can be reached by adding particles to a recurrent configuarations without doing any toppling.

$$\underbrace{\underbrace{\begin{bmatrix}0&1&3\\1&2&1\\2&3&3\end{bmatrix}}_{\text{in } R} \xrightarrow{a_9} \underbrace{\begin{bmatrix}0&1&3\\1&2&1\\2&3&4\end{bmatrix} \xrightarrow{a_3} \begin{bmatrix}0&1&4\\1&2&1\\2&3&4\end{bmatrix} \xrightarrow{a_1} \begin{bmatrix}1&1&4\\1&2&1\\2&3&4\end{bmatrix}}_{\text{unstable}}}_{\text{all in } \mathcal{A}}$$

Step 1:

If two configs $(c_1, c_2) \in \mathcal{A}$ when relaxed reaches same config in R, then they are said to be equivalent $c_1 \Leftrightarrow c_2$.

To express it precisely, define toppling operators

$T_i\, c = c'$ which topples an unstable site i.

If $c = \{z_i\}$ then

$$T_i\, c = \begin{cases} z_j - \Delta_{ij} & \text{if } \quad z_i \geq \Delta_{ii} \\ z_j & \text{otherwise.} \end{cases}$$

Then $c_1 \Leftrightarrow c_2$ if and only if there exist sequences

$$T_{i_1} T_{i_2} \cdots T_{i_n}\, c_1 = T_{i'_1} T_{i'_2} \cdots T_{i'_m}\, c_2$$

$$\Rightarrow \quad z_j - \sum_{k=1}^{n} \Delta_{i_k j} = z'_j - \sum_{k=1}^{m} \Delta_{i'_k j}$$

total number of nodes.

$$\Rightarrow \quad \boxed{z'_j - z_j = \sum_{k=1}^{m} \Delta_{i'_k j} - \sum_{k=1}^{n} \Delta_{i_k, j} = \sum_{i=1}^{N} g_i\, \Delta_{ij}}$$

$$\Rightarrow \boxed{z'_j - z_j = \sum_{k=1} \Delta_{i_k j} - \sum_{k=1} \Delta_{i_k, j} = \sum_{i=1} g_i \Delta_{ij}}$$

multiplicity (integer)

Step 2: Let $\hat{e}_i$ are unit vectors in N-dimensional space.

A config $c \equiv \{z_i\}$ correspond to a vector $\vec{z} = \sum_{i=1}^{N} z_i \hat{e}_i$

If we define a set of N vectors

$$\vec{n}_i = \sum_{j=1}^{N} \Delta_{ij} \hat{e}_j$$

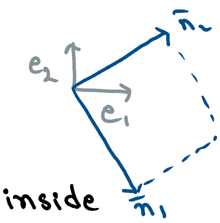
then this, in vector notation means

$$\vec{z}' - \vec{z} = \sum_{i=1}^{N} g_i \Delta_{ij} \hat{e}_j = \sum_{i=1}^{N} g_i \vec{n}_i$$

This means

$$\boxed{c' \Leftrightarrow c \quad \text{if} \quad \vec{z}' - \vec{z} \in \text{span of } \{\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N\}}$$

Step 3: Then all elements of an equivalent class can be generated from only one $\vec{z}$ in the unit cell of the space spanned by $\{\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N\}$.



Total number of equivalent classes = number of $\vec{z}$ vectors inside the unit cell.

$\parallel$

total number of configs in R $=$

Step 3: volume of the unit cell $= \det \Delta$

because $\vec{n}_i = \sum_j \Delta_{ij} \hat{e}_j$

Jacobian of the transformation

Finally gives $\boxed{\|R\| = \det \Delta}$

(3) Correlations can also be expressed in terms of toppling matrix.

G_{ij} = average number of toppling at a site j in an avalanche started by a_i (adding grain at a site i)

It is related to Δ_{ij}. How?

At jth site, during an avalanche

$$\underbrace{\sum_{k\neq j} G_{ik}(-\Delta_{kj}) + \delta_{ij}}_{\text{average influx}} - \underbrace{G_{ij}\Delta_{jj}}_{\text{average outflux}} \simeq 0$$

$$\Rightarrow \boxed{\sum_{k=1}^{N} G_{ik}\Delta_{kj} = \delta_{ij}} \Rightarrow \boxed{G_{ij} = (\Delta^{-1})_{ij}}$$

For nearest neighbor toppling on a hyper cubic lattice in d-dimension

discrete laplacian $\leftarrow$ $\Delta G(\vec{r}) = \delta_{\vec{r},0}$

$\Rightarrow$ for large distance $G(r) \sim 1/r^{d-2}$

Remark: Following this line of calculation many exponents have been derived.

(1) height-height correlation $\sim r^{-2d}$ Majumdar and Dhar' 91

(2) average number of toppling on $L\times L$ square grid $\sim L^2$ Dhar' 1990.

(3) On a tree (Bethe lattice) Dhar & Majumdar' 1990

$P(n) \sim n^{-3/2}$ prob of total n topplings.

$P(t) \sim t^{-2}$ duration of an avalanche.

(4) Prob of distinct sites toppled in an avalanche

$P(s) \sim s^{-\tau}$ with $\tau \simeq 5/4$

Priezzhev et. al. 1996, using connection to spanning tree.

Pietronero et al 1994, using renormalization group calculation.

How to test if a configuration is recurrent?

(1) Multiplication test : take square grid and nearest neighbor toppling

2	3	2
3	0	0
1	2	3

Algebra

$$a_i^4 = \begin{cases} a_{i_1} a_{i_2} a_{i_3} a_{i_4} & \text{bulk sites} \\ a_{i_1} a_{i_2} a_{i_3} & \text{boundary} \\ a_{i_1} a_{i_2} & \text{corner sites} \end{cases}$$

$$\prod_{i=1}^{N} a_i^4 = \prod_{i=1}^{N} a_i^{n_i}$$

$n_i :=$ number of neighbors of ith site.

On a recurrent space $\mathcal{R}$ inverse of a_i exist.

$$\Rightarrow \boxed{\prod_{i=1}^{N} a_i^{4-n_i} = \mathbb{1}}$$

This means, in a configuration, if we add 1 grain at the boundary (2 at corner) and then relax and it gives back same configuration then it is recurrent.

The burning test : If Δ is a symmetric matrix, there is an algorithm.

(a) start with the config $\mathcal{C}$ that we want to test.

2	3	2
3	0	0
1	2	3

(a) Add particles at the boundary (1 and 2).
At this stage all sites are "unburnt".

4	4	4
4	0	1
3	3	5

(b) "Burn" the sites with $z_i >$ number of "unburnt" neighbors.
Do this by simultaneous update.

	0	1
	3	

(c) Repeat recursively untill fire stops.

(d) If all sites are burnt then $\mathcal{C}$ is recurrent.

	0	1

Not recurrent.

Stationary state : State of a system $|P(t)\rangle = \sum_{\mathcal{C}_s} P_t(\mathcal{C}_s) |\mathcal{C}_s\rangle$

Time evolution

$$|P(t+1)\rangle = W |P(t)\rangle \quad \text{with } W = \frac{1}{N} \sum_i a_i$$

$W \equiv |\mathcal{R}| \times |\mathcal{R}|$

let $|\lambda\rangle$ are eigenvector of W corresponding to eigenvalue λ.

$$\Rightarrow \quad |P(t)\rangle = W^t |P(0)\rangle = \lambda_0^t |\lambda_0\rangle \cdot A_0 + \lambda_1^t |\lambda_1\rangle A_1 + \dots + \lambda_R^t |\lambda_R\rangle A_R$$

with $A_n = \langle \lambda_n | P(0)\rangle$

Largest eigenvalue $\lambda_0 = 1$ gives the stationary state

$$\boxed{|P_s\rangle = |\lambda_0\rangle}$$

Q: How to find eigenstates $|\lambda\rangle$ of W?

$[a_i, a_j] = 0 \Rightarrow [a_i, W] = 0$ means all have a common set of eigenvectors $|\lambda\rangle$

let

$$\boxed{a_j |\lambda\rangle = e^{i\phi_j} |\lambda\rangle}$$

$$a_i^{\Delta_{ii}} = \prod_{j \neq i} a_j^{-\Delta_{ij}} \Rightarrow \boxed{\prod_j a_j^{\Delta_{ij}} = \mathbb{1}} \quad \text{using that } a_j^{-1} \text{ exist.}$$

Combining we get

$$e^{i\sum_{j=1}^{N} \Delta_{ij}\phi_j} = 1 \Rightarrow \boxed{\sum_j \Delta_{ij}\phi_j = 2\pi m} \quad \text{with } m = 0, 1, \dots, R-1$$

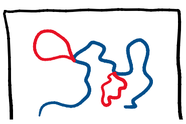
$$\Rightarrow \boxed{\phi_j = 2\pi^{m} (\Delta^{-1} \cdot)_j} \quad \text{gives eigenvalues.}$$

Still to solve: (1) Prob distribution of avalanch size for 2d lattice.

(2) Upper critical dimension $d_c = 4$. Above which Bethe lattice (tree) solution gives the exponents.

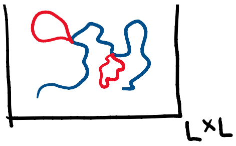
Other models of SOC:

① Loop erased random walk.



Reaches a steady state with probability of erased length of loop

$$P(\ell) \sim 1/\ell^{\delta} .$$



Reaches a steady state with probability of erased length of loop $P(\ell) \sim 1/\ell^{\delta}$.

ASM and Potts model, spanning tree.

Generating function of number of recurrent configurations with a FIXED number of total grains relates to partition function of Potts model in $q \to 0^+$ limit

(i) q-state Potts model.

[Next class ...]

Abelian group, Identity element, patterns.