Fourth Lecture

Monday, August 16, 2021 3:23 PM

Discrete holomorphic functions, tropical algebra, and Apollonian circle packing.

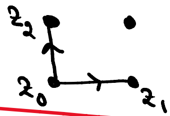
What is a discrete analogue of $z^{p/q}$ on complex plane?

Discrete holomorphic function on many sheeted Riemann surface.

Holomorphic function	Discrete holomorphic funcion on $\mathbb{Z}^2$

Cauchy-Riemann condition

$$f(z) = u(x,y) + i\,v(x,y)$$



$z = m\cdot\epsilon + in\epsilon$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = i\,\frac{\partial f}{\partial x} \quad \longrightarrow \quad \boxed{f(z_2) - f(z_0) = i\big(f(z_1) - f(z_0)\big)}$$

Then $f(z)$ is discrete holomorphic.

For further readings see

(1) Book by L. Lovász in Discrete analytic functions: an exposition

(2) Duffin, Duke math J, 23, 335 (1956).

(3) Mercat, Commun. Math. Phys, 218, 177 (2001).

What are discrete holomorphic functions on multi-sheeted Rieman surface?

[Sadhu, Dhar, PRE 85, 021107 (2012)] ← see Appendix.

More specificaly, what are discrete analoge of z^n ?

For positive integer n, examples of discrete holomorphic functions are

$$z,\; z^2,\; \left(z^3 + \epsilon^2 \frac{\bar{z}}{2}\right),\; \left(z^4 + 2\epsilon^2 z\cdot\bar{z}\right), \cdots$$

More generally,

$$F_n(z,\epsilon) = z^n\left[1 + \frac{\epsilon^2}{z^2}\, g_n^{(1)}\!\left(\frac{\bar{z}}{z}\right) + \frac{1}{2!}\cdot\frac{\epsilon^4}{z^4}\, g_n^{(2)}\!\left(\frac{\bar{z}}{z}\right) + \cdots\right]$$

with

$$g_n^{(1)}(x) = \frac{2}{n-3}\, B(n,4)\, x$$

$$g_n^{(2)}(x) = \frac{7!}{(4!)^2}\cdot\frac{2^2}{n-6}\cdot B(n,7)\cdot x^2$$

$$g_n^{(3)}(x) = \frac{10!}{(4!)^3}\cdot\frac{2^3}{n-9}\cdot B(n,10)\, x^3 - \frac{6^3}{n-7}\, B(n,8)\, x$$

higher terms can be constructed from the CR condition.

The construction satisfies

$$\lim_{\epsilon\to 0} F_n(z,\epsilon) = z^n \qquad \text{and} \qquad F_n(z,\epsilon) = a^n F_n\!\left(\frac{z}{a}, \frac{\epsilon}{a}\right)$$

Analytical continuation of $f_n(z,\epsilon)$ to rational numbers gives an analogue of $z^{p/q}$.

How are patterns selected? For a given background, how is the pattern "selected" amongst many possible patterns.

Deepak's brilliant idea

A least "action" principle (a lazy man's maxim)

Ref: Sadhu and Dhar, J stat mech, 2011.

Statement: The actual pattern is the stable pattern reached by minimum number of toppling.

Let $\tilde{T}_N(\vec{x}) : \mathbb{Z}^d \to \mathbb{N}$ — includes the N grains added at the centre.

Then $T_N := \min \{ \tilde{T}_N \mid \Delta \tilde{T}_N + z_0 < z_c \}$ $\qquad z_0 : \mathbb{Z}^d \to \mathbb{N}$

where the minimum is taken point wise.

The final pattern is

$$z_N = \Delta T_N + z_0$$

An equivalent statement (Fey, Levin, Peres): If $z_0 : \mathbb{Z}^d \to \mathbb{N}$ and $\tilde{T}_N : \mathbb{Z}^d \to \mathbb{N}$ satisfy $z_0 + \Delta \tilde{T}_N < z_c$, then z_0 is stabilizable, and the actual toppling function T_N satisfies $T_N \le \tilde{T}_N$ in pointwise sense.

Arguement: The relaxation dynamics is deterministic. So, there is only one T_N function.
For $\tilde{T}_N(x,y)$ we need to relax the toppling conditions.
Define,

(1) legal sequence: sequence of topplings in which only unstable sites topple.

(2) Stabilizing sequence: any sequence of topplings that lead to a final stable configuration.

$$\Delta \tilde{T}_N + z_0 < z_c$$

This also includes toppling at stable sites.

(3) A legal sequence is a subsequence of stabilizing sequence.
This is simply because an unstable site can not stabilize by toppling at other sites.

(4) Because of Abelian nature, there is a unique legal stabilizing sequence. T_N is the corresponding function.

(5) This means, given a z_0,

$$T_N(x) \leq \tilde{T}_N(x)$$

What is the use of this least "action" principle?

Application 1:

This variational principle gives us a way to compare different trial patterns and select the pattern corresponding to the minimum toppling for the same background, which is the actual pattern.

The set of patterns over which optimization is to be performed is large. However, one can make a trial pattern close to the actual pattern (say, by enlarging a small pattern, or choosing a close enough background-pattern) and then optimise the T_N.

[Ref: Friedrich et al, "Fast simulation of large scale growth model"]

Falguni Pathak, MSc thesis supervised by Deepak

Application 2:

The weak convergence of ASM pattern was derived [Pegden, Smart] by identifying the continuum limit of this least "action" principle using viscosity solution theory.

Let $\psi(x) = -\frac{1}{2\pi}\log|x|$ and $\omega \in C(\mathbb{R}^2)$. Then the asymptotic re-scaled toppling function is point wise minimum of $\phi = \psi + \omega$ with constraints

$$\phi_\infty = \min\left\{ \phi \;\middle|\; \phi \geq 0 \text{ and } D^2\phi \in \bar{\Gamma} \right\}$$

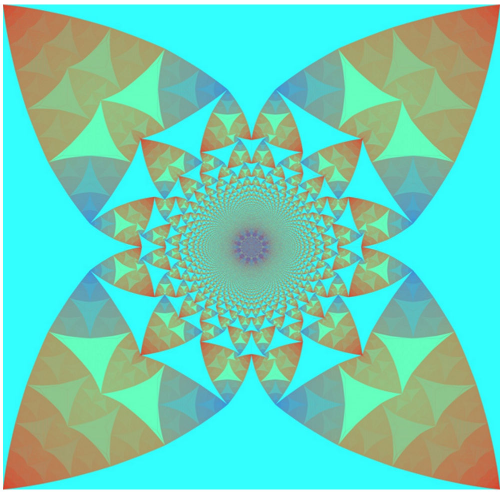
here $\bar{\Gamma}$ is a subset of 2×2 real symmetric matrices that was determined by Levin, Pegden and Smart. See our discussion below about connection to Apollonian Circle Packing.

Back to the square grid ASM pattern

Relation to Apollonian circle packing problem.

[Ref: Levin, Pegden, and Smart in Geom Funct Anal, 26 (2016) 306
"Apollonian structure in the Abelian sandpile"]

[Ref: Levin, Pegden, and Smart in Geom Funct Anal, 26 (2016) 306
"Apollonian structure in the Abelian sandpile"]



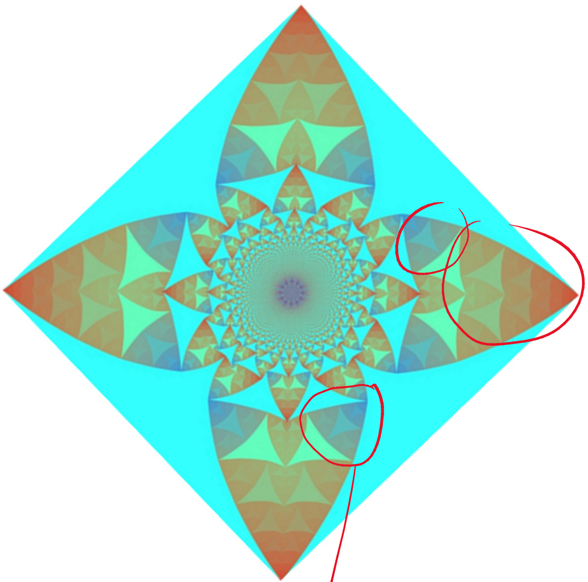
Difficulties:

(1) There are possibly infinite number of periodic patches with $\rho - \rho_0 = \frac{1}{l}$ ← unit cell volume.
[Ostojic]

(2) Patch boundaries are not simple.

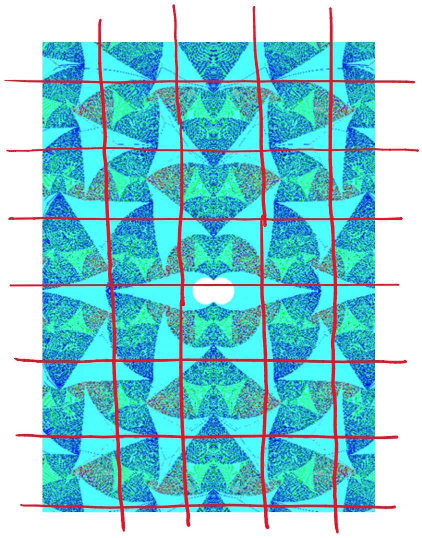
(3) Adjacency graph is not simple.

Rotate



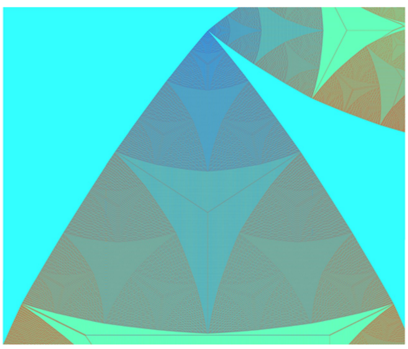
Adjacency graph for patches

$\xrightarrow{1/2^2}$

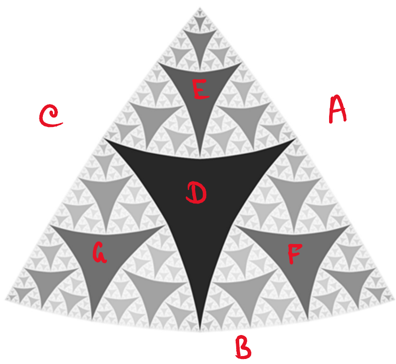


The cyan patches seems to lie on a squre grid.

How are rest of the patches distributed?



in gray scale



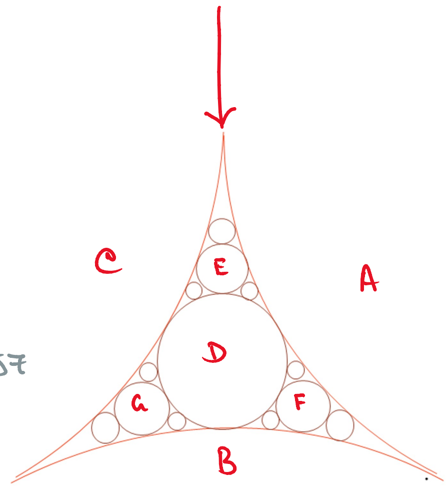
Taken from the Levin et al Paper.

Apollonian circle packing.

Remark: fractal dimension of ACP is still an open problem.

See Thomas and Dhar, J Phys A 27 (1994) 2257

Housdorf dim $d_f = 1.305686729$



64

An important feature that immediatly comes out from this observation is that neighobouring patches touch only at a point. The patch bundaries are the limiting curves formed by these points.

The connection is even stronger.

For each circle in ACP there are three parameters

$$(x-a)^2 + (y-b)^2 = (c-2)^2$$

Construct a 2x2 real symmetric matrix

$$M(a,b,c) = \begin{pmatrix} c+a & b \\ b & c-a \end{pmatrix}$$

The patch corresponding to the circle has scaled toppling function

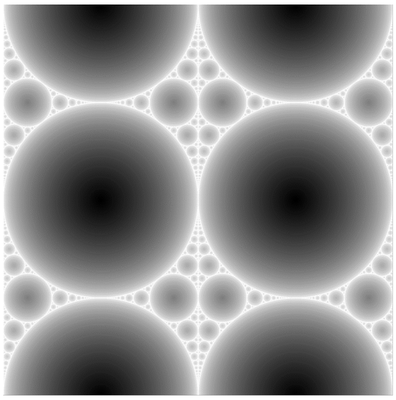
$$\phi(r) = \frac{1}{4} r^t M r + D r + f$$

So, given M for out-side patches in a leaf, we can determine M for all inside patches from the ACP correspondance.

Characterization of all M's in the pattern. Use the least action principle discussed above.

The set $\bar{\Gamma}$ is the union of downward cones whose peaks are the set of 2x2 real symmetric matrices Γ from the Apollonian band packing

Ref: Levin, Pegden, and Smart in Geom Funct Anal, 26 (2016), 306



What is downward cone?

For $A, B \in S_2$, we say $B \leq A$ if $A-B$ is non-negative definite. Then

$$\bar{\Gamma} := \{ B \mid B \leq A \text{ for some } A \in \Gamma \}$$

Relation to Tropical curve and ASM pattern

Tropical algebra: Define a new "addition" and "multiplication" rule.

[Ref: Speyer and Sturmfels, Math. mag. 82, 163 (2009).]

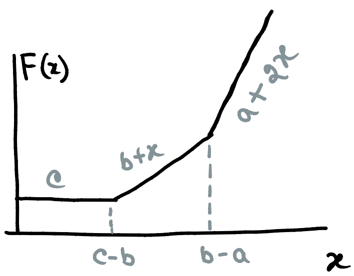
$$a \oplus b = \max\{a, b\} \quad \text{and} \quad a \otimes b = a + b.$$

for $a, b \in \mathbb{R}$

Familiar properties exist for this algebra: commutativity, associativity, identity, and distributive.

Tropical polynomials

$$F(x) = a \otimes x^2 \oplus b \otimes x \oplus c$$
$$= \max\{a + 2x, bx, c\}$$



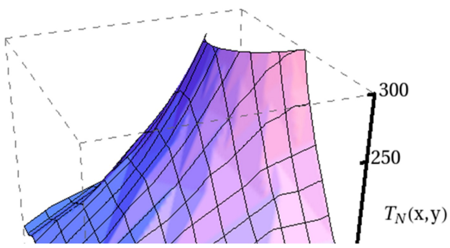
A tropical polynomial is a piecewise linear function that is also convex.

Their 2d generalization.

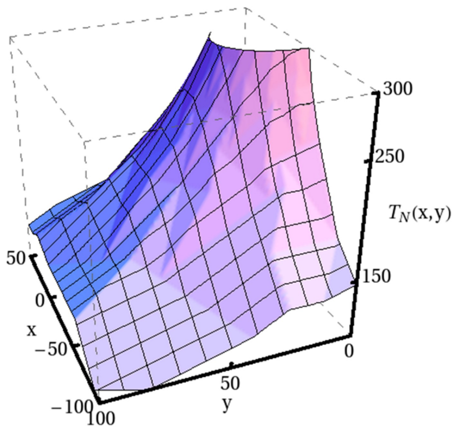
Let $A \in \mathbb{Z}^2$ be any finite set. Then

$$\text{Trop } F(x,y) = \max_{(i,j)\in A} \{ a_{ij} + ix + jy \} \longleftarrow \text{analogue of } \sum_{(i,j)\in A} a_{ij} x^i y^j$$

What are their connection to sandpile pattern?



Remember the toppling function in triangular pattern? The surface is formed of piecewise planes. However, a careful inspection shows that the surface is not convex, therefore not a tropical polynomial.



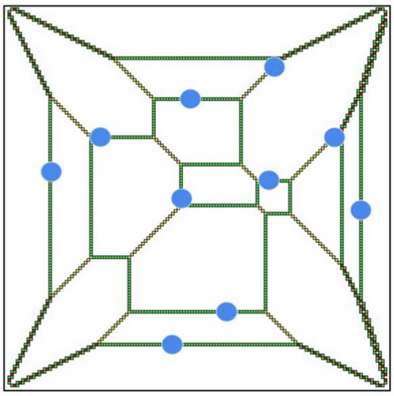
Remember the toppling function in triangular pattern? The surface is formed of piecewise planes. However, a careful inspection shows that the surface is not convex, therefore not a tropical polynomial.

Connection to tropical curves come in a subtle way.

A connection to sandpile pattern.

[Ref: Kalinin, Guzmán-Sáenz, Prieto, Shkolnikov, Kalinina, Lupercio in PNAS, 115, (2018) E8135]

[Ref: Kalinin, in Frontiers in Physics "Pattern formation and tropical geometry"]



Pattern generated in a finite grid with sink at the boundary with initial heights

$$z_0 = \begin{cases} 4 & \text{at blue dots} \\ 3 & \text{elsewhere} \end{cases}$$

Color code: white (3), green (2), yellow (1), red (0).

The lines form <u>Tropical curves.</u>

Tropical curves:

The geometric counterpart of the tropicalization is as follows. Given a complex algebraic curve $\mathscr{C}_t$ defined by a polynomial

Taken from Kalinin PNAS paper.

$$F_t(x,y) = \sum_{(i,j)\in\mathscr{A}} \gamma_{ij} t^{a_{ij}} x^i y^j = 0, \left|\gamma_{ij}\right| = 1,$$

we call the amoeba A_t the image of $\mathscr{C}_t$ under the map $\log_t(x,y) = \left(\log_t|x|, \log_t|y|\right)$, $A_t := \log_t(\mathscr{C}_t)$. The limit of the amoebas A_t as $t \to +\infty$ is called $\mathrm{Trop}(\mathscr{C})$, the tropicalization of $\mathscr{C}_t$.

The limit $\mathrm{Trop}(\mathscr{C})$ can be described entirely in terms of the tropical polynomial $\mathrm{Trop}(F)$ (**Eq. 1**). This fact can be proved by noticing that on the linearity regions of $\mathrm{Trop}(F(x,y))$, one monomial in F_t dominates all of the others, and therefore, F_t cannot be zero, and consequently, we conclude that the limit $\mathrm{Trop}(\mathscr{C})$ is precisely the set of points (x,y) in the plane where the (3D) graph of the function

$$\mathrm{Trop}(F(x,y)) = \max_{(i,j)\in\mathscr{A}} \left(a_{ij} + ix + jy\right)$$

is not smooth. This set of points is known as the corner locus of $\mathrm{Trop}(F(x,y))$. For this reason,

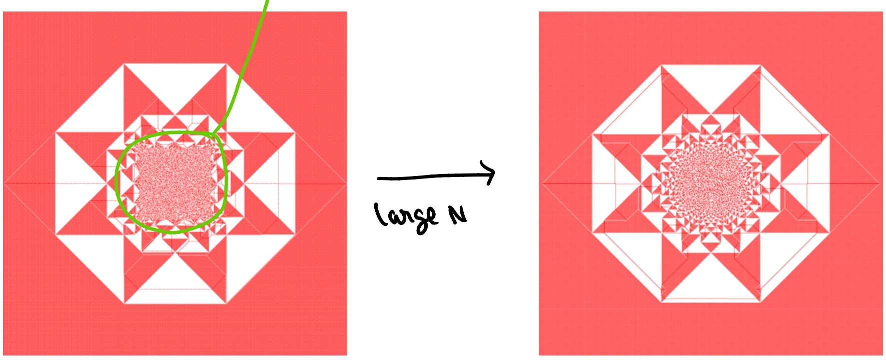
Remark: inside the white regions rescaled toppling function ϕ is linear.

Question: Do the patch boundaries in triangular lattice pattern form tropical curves?

Robust ness of patterns (only by pictures)

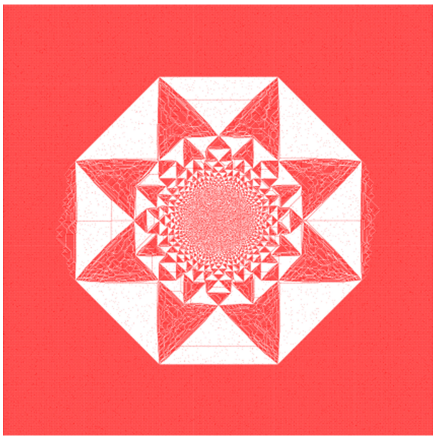
[Ref: Sadhu and Dhar, J Stat Mech (2011) P03001]

Random addition in side a region

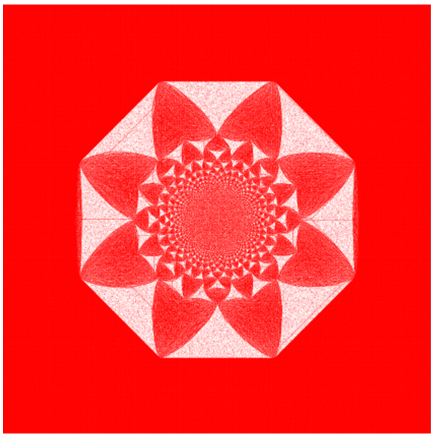


For large N, relative size of region of addition decrease and one gets back the asymptotic pattern of single addition site.

Noisy back ground (1's replaced by 0's at random sites)

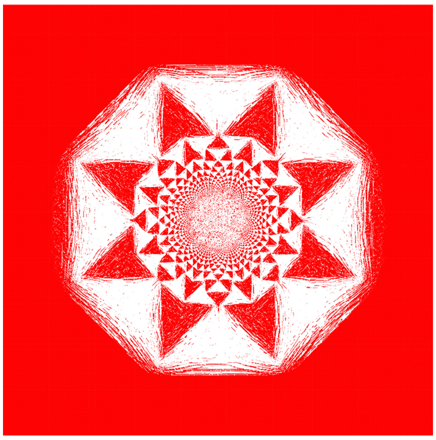


1% noise

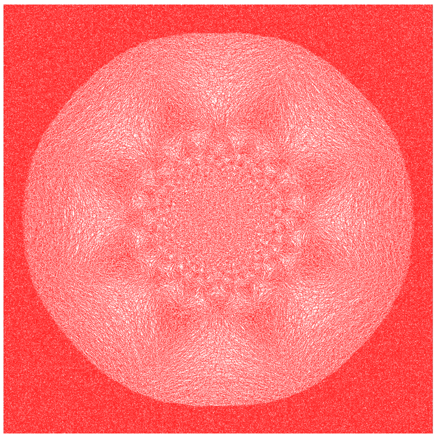


10% noise

Background with 0's replaced by 1's randomly

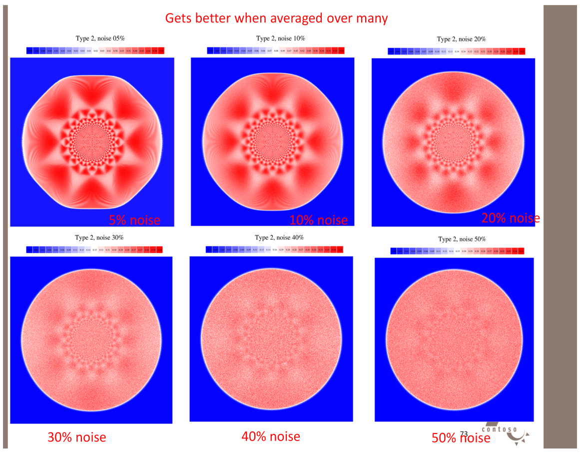


1% noise

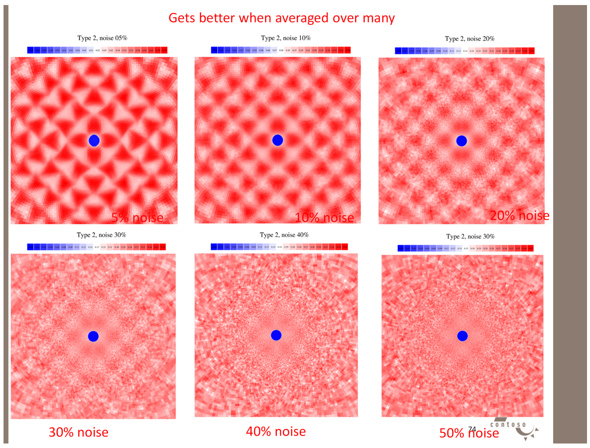


10% noise

Pattern gets better resolved when averaged over noise realizations.



their $1/\bar{z}$ transformation



A few open problems I am interested in.

(1) A full characterization of the square grid pattern.

(2) Pattern characterization in 3d.

(1) A full characterization of the square grid pattern.

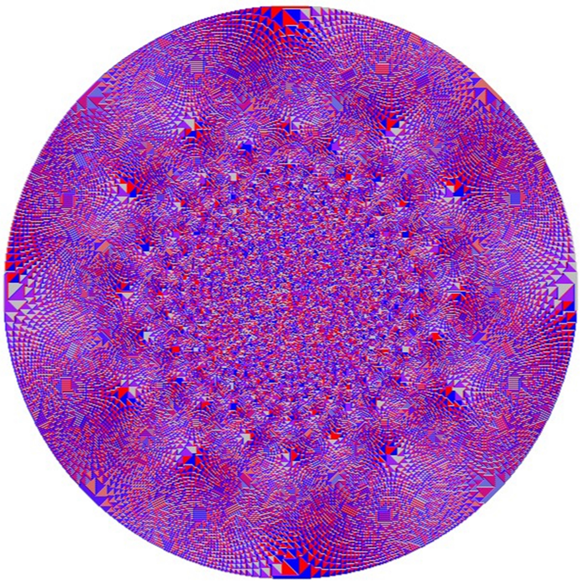
(2) Pattern characterization in 3d.

(3) Identity for F-lattice, triangular lattice, and square grid.

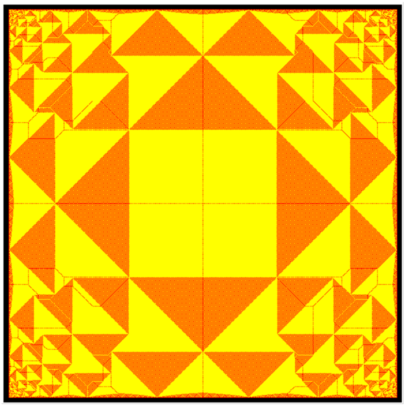
(4) Robustness of pattern using least "action" principle.

(5) Relation to tropical curves?

(6) Eulerian Walker pattern [Rahul Dandekar and Deepak Dhar]



Eulerian Walker Pattern



Finite grid F-lattice pattern