Second lecture

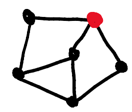
Sunday, August 1, 2021 8:15 PM

Stable configs, recurrent states, etc in a Mathematical language

[from the AMS article "what is a sandpile?" by Levin & Propp]

[Deepak Dhar, review in Physica A, 369 (2006), 29
Dhar, Ruelle, Sen, Verma, J Phys A, 28 (1995) 805]

- G : a finite connected graph with one sink node.
- V : set of non-sink nodes.
- $\mathbb{Z}^V$: the free Abelian group on V.



- M : set of all stable configurations.
- Δ_{ij} : toppling matrix on V.

$$\Delta_{ij} = \begin{cases} -d_i & \text{if } i=j \\ 1 & \text{if } i,j \text{ are neighbors} \\ 0 & \text{else.} \end{cases}$$

d_i degree of vertex i.

Δ is the reduced Laplacian of G (reduced because it excludes the sink nodes)

- a_i : addition on M + relaxation.

commutation $[a_i, a_j] = 0$.

This gives M the structure of a commutative Monoid.

- Minimal ideal of M are called recurrent.
- Minimal ideal of a finite commutative monoid is always a group.

$\boxed{\text{The Sandpile group } K(G) \text{ is the minimal ideal of } M.}$

The group $K(G)$ is an isomorphism invariant of the graph G and it is independent of the choice of sink up to isomorphism.

We shall discuss this more later, today.

- Two vectors $c_1, c_2 \in \mathbb{Z}^V$ are equivalent if and only if their difference lies in the $\mathbb{Z}$-linear span of vectors Δ_i

Each equivalence class in $\mathbb{Z}^V$ contains exactly one recurrent element.

$$K(G) = \mathbb{Z}^V / \Delta \mathbb{Z}^V$$

- Index of the subgroup $\Delta\mathbb{Z}^V$ is $\det\Delta$ which also the order of $K(G)$.

Remarks: By matrix tree theorem $\det\Delta$ is also the number of spanning trees on G.

[On G this number is det of graph Laplacian (adjacency matrix) with any arbitrary row and column deleted. Δ is the graph Laplacian when

[On G this number is det of graph Laplacian (adjacency matrix) with any arbitrary row and column deleted. Δ is the graph Laplacian when row-column of sink node are deleted]

There is a one-to-one correspondence between a spanning tree on G and a recurrent configuration in ASM.
For an explicit connection to spanning tree, see later discussions.

Remark: Number of spanning trees on G equals to $T(1,1)$ where $T(x,y)$ is Tutt polynomial on G.

There is an even stronger connection. [Merino López]

$$\sum_{C \in \text{Recurrent}} y^{|C|+\delta-e} = T(1,y)$$

here, $|C| \equiv$ total number of grains in C, $e \equiv$ number of edges on G, and $\delta \equiv$ degree of the sink vertex.

For more details see later discussions.

Remarks: The sandpile group gives algebraic manifestations to many classical enumerations of spanning trees. For example, Caley's formula for the number of spanning trees on complete graph K_n becomes

$$K(K_n) = n^{n-2} \equiv (\mathbb{Z}_n)^{n-2}$$

and on complete bipartite graph

$$K(K_{m,n}) = m^{n-1} n^{m-1} \equiv \mathbb{Z}_{mn} \times (\mathbb{Z}_m)^{n-2} \times (\mathbb{Z}_n)^{m-2}.$$

Remarks: In analogies between graphs and algebraic curves, sandpile group is known by different names "group of components", "Jacobian group", and "critical group".

Quote from the paper: "A deep analogy between graphs and algebraic curves can be traced back implicitly to a 1970 theorem of Raynaud, which relates the component group of the Neron model of the Jacobian of a curve to the Laplacian matrix of an associated graph. In this analogy, the sandpile group of the graph plays a role analogous to the Picard group of the curve"

What is the sandpile group? a simple version.

[Ref: Michael Creutz, Computers in Physics 5, 198 (1991)
Deepak Dhar, review in Physica A, 369 (2006), 29
Dhar, Ruelle, Sen, Verma, J Phys A, 28 (1995) 805]

The operators a_i when acted on the recurrent config generate a finite Abelian group.

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$$A = \prod_{i\in V} a_i^{m_i}$$ are group elements.

because by $a_i^{\Delta_{ii}} = \prod_{j\neq i} a_j^{-\Delta_{ij}}$ we can reduce the power $m_i < \Delta_{ii}$.

Using Abelian property

(1) Inverse $\quad A^{-1} = \prod_{i\in V} a_i^{-m_i}$

(2) Identity $\quad \mathbb{1} = \prod_{i\in V} a_i^{-r_i+\Delta_{ii}} = \prod_{i\in V} a_i^{\Delta_{ij}}$ for all j

r_i is number of non-sink neighbors.

(3) Commutativity $\quad [A,B]=0$

Order of the group $|K(G)| = \det \Delta$.

An isomorphism: $\quad C_1 \oplus C_2 = C_3 \quad$ for stable configurations

where $\oplus$ means adding the heights of the two config at respective nodes and then relaxing.

Clearly if either C_1 or C_2 is recurrent, the C_3 is recurrent.

$\boxed{\text{Under operation } \oplus \text{ elements in } \mathcal{R} \text{ (recurrent set) form an Abelian group isomorphic to the algebra generated by } a_i \text{ acting on } \mathcal{R}.}$

This is easy to see by noting that $C \leftrightarrow \prod_i a_i^{z_i}$.

(i) $\oplus$ is associative and Abelian.

(ii) $\prod_i a_i^{-r_i+\Delta_{ii}} \leftrightarrow I \quad$ is the identity configuration.

(iii) for finite abelian group $a_i^{|R|} = 1$

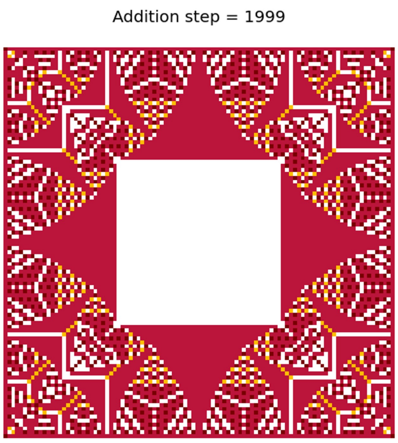
$$\Rightarrow C^{-1} = \underbrace{C \oplus C \oplus \cdots \oplus C}_{|R|-1 \text{ times}}$$

$\boxed{\text{The config } C \text{ isomorphic to } a_i \text{ is } a_i I}$.

A corollary: the identity configuration gives a way to test if a config C is recursive or not.

$$\boxed{C \oplus I = C} \text{ if and only if } C \in \mathcal{R}.$$

How does the identity configuration look?



color code:

black = 0
yellow = 1
white = 2
Red = 3

100×100 square grid.

Imp: ignore the boundary sites (they are artifacts of my simulation).

An algorithm to generate the identity configuration I.

use $\prod_{i \in V} a_i^{-\pi_i + \Delta_{ii}} = \mathbb{1}$ on recurrent space.

Operators	$\longleftrightarrow$	configurations
$\prod_{i \in V} a_i^{-\pi_i + \Delta_{ii}} = \mathbb{1}$	$\longleftrightarrow$	$I_0 \equiv \{ z_i = -\pi_i + \Delta_{ii} \}$

because empty sites in centre

Not recurrent, and that's why it is not identity element I.

Take any $C \in R$

$\mathbb{1} C = C \longleftrightarrow I_0 \oplus C \neq C$

$\mathbb{1}^n C = C \longleftrightarrow I_n \oplus C$ where $I_n = \underbrace{I_0 \oplus \cdots \oplus I_0}_{n\text{-times}}$

If n is sufficiently large, $I_n \in R$ and then

$I_n \equiv I$ and $I_n \oplus C = C$.

We will show how to characterise the intricate structure of the identity pattern.

Before that, an interesting fact.

- Let $f_i(c_1, c_2) :=$ number of toppling during the operation $c_1 \oplus c_2$.

Then $\boxed{f_i(I, c) \text{ is same for all } c \in R.}$

Other Interesting facts about ASM

- Relation to Potts model: [Dhar, Physica A, 369 (2006), Section 7.2]

On a connected graph G, spin variables $\sigma_i = \{1, 2, \cdots, q\}$ on each nod, with probability measure

$$P(\{\sigma_i\}) = \frac{1}{Z} \cdot e^{\sum_{\langle ij \rangle} J \, \delta_{\sigma_i,\sigma_j}}$$ ← Kronecker delta.

Special case $q=2$ is the Ising model on G.

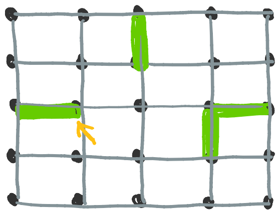
Graphical representation:

Partition function $$Z(q) = \sum_{\{\sigma_i\}} e^{\sum_{\langle ij\rangle} J \, \delta_{\sigma_i \sigma_j}}$$

$$= \sum_{\{\sigma_i\}} \prod_{E(G)} \left[1 + v \, \delta_{\sigma_i \sigma_j} \right]$$

$v \to e^J - 1$

$$= \sum_{E' \subset E(G)} q^{c(E')} \, v^{|E'|}$$



$c = N-4$
$|E'| = 4$

Here, $E' \equiv$ configuration edges on G.
$c(E') \equiv$ number of disconnected clusters in E'
(an isolated site is a single cluster)
$|E'| \equiv$ number of edges in E'

Interesting limits:

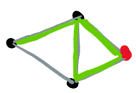
(i) $J \to -\infty$: $Z(q)$ is the chromatic polynomial of G.
(number of distinct ways nodes can be colored by q colors so that no two neighbors have same color)

(ii) $q \to 1$: $Z(q)$ gives generating function of configs in bond percolation with $p = \frac{v}{1-v}$
[Wu, J. Stat. Phys., 18, 1978]

(iii) $q \to 0+$: relates to spanning tree and ASM.

First: relation to ASM on G with N nodes with One sink node.

For $q \to 0^+$



$$Z(q) = \sum_{E' \subset E(G)} q^{c(E')} \, v^{|E'|}$$

$$= q \, v^{N-1} H(v) + \text{higher order in } q$$

↳ a polynomial of maximum degree $E-N$

To relate to ASM, consider $C \in R$ and $m(C) =$ total number of sandgrains

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To relate to ASM, consider $C \in R$ and $m(C)$ = total number of sandgrains in G.

Then

$$F(y) = \sum_m \underbrace{g_m}_{\text{number of } C\in R \text{ with } m \text{ grains}} \cdot y^m = y^{N-1+E-\underbrace{E_s}_{\text{degree of sink node}}} H(v=y-1)$$

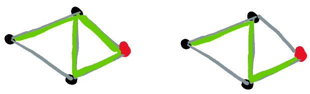
Ref: (1) Merino, Ann Comb, 1 (1997) 253.

(2) Cori, Borgne, Adv App Math, 30 (2003), 44.

The proof uses the burning algorithm. Burning starts at sink site and invades the bulk. The time to burn a site i is related to the minimum links to reach that site among the subgraphs E'. Then the height z_i is constructed from this burning time using additional rules.

See, Dhar, Physica A (2006), section 7.2 for details.

Second: relation spanning tree.



spanning tree.

$$Z(q) = \underbrace{q\, v^{N-1} H(v)}_{\text{Includes tree and loops.}} + O(q^2) \qquad \text{for } q \to 0^+$$

To single out the spanning trees, take additional $v \to 0$ limit (high temperature expansion in Physics).

In general, for Potts model with $v_{ij} = e^{J_{ij}} - 1 = \beta w_{ij}$

$$\lim_{\beta\to 0} \frac{1}{\beta^{N-1}} \lim_{q\to 0^+} \frac{Z(q)}{q} = S(G)$$

sum of weights of all spanning trees on G.

There is a one-to-one correspondence between a spanning tree on G and a recurrent configuration in ASM.

Remarks: Relation of ASM and Potts model leads to many exact results of critical exponents for the critical state. In 2d, $q \to 0$ Potts model corresponds to a conformal field theory with central charge $c = -2$.

[Saleur and Duplantier, (1987)]

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For example, average path length in a spanning tree in 2d

$\sim r^{5/4}$ where r is the Euclidean distance

Then, the map gives the average time for an avalanche in 2d-ASM to spread a distance r scales as $\sim r^{5/4}$.

Remarks: few more exact results about probability of heights in stationary state.

(i) On 2d square grid $P(z)$ in bulk sites in $L\to\infty$ limit is know.

Majumdar & Dhar (1991), Priezzev (1994)

(ii) Two-point correlation of height is know in 2d using logarithmic conformal field theory [Piroux & Ruelle, 2005]

$$P(z_i=a, z_j=b) \sim f_a f_b + \frac{1}{r_{ij}^4}\left(A_{ab}\left[\log r_{ij}\right]^2 + B_{ab}\log r_{ij} + C_{ab}\right)$$

Remark: There are many variants of sandpile model.

(1) Continuous height model: (a) $z_i \geq 0$, (b) threshold z_c, (c) in toppling a site is fully emptied and equally distributed among neighbors

(d) driving is by adding a random amount of $z\in[a,b)$

What is surprising is that probability distribution of height z gets peaked around discrete values and their widths decreas with increasing system size. The conjecture is that, critical behaviour is same as ASM, although Abelian property is lost. [Sadhu & Dhar, PRE 2009]

(2) Stochastic sandpile: Manna model and its generalization.

Difference from BTW model is that in toppling particles are randomly distributed among neighbors.

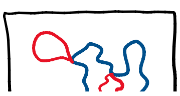
Abelian property is retained but inverse of a_i does not exist.

So the a_i operators form an Abelian semi-group.

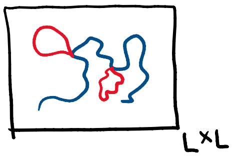
All stable configs are recurrent and their probability is NOT uniform.

Behavior is different from BTW model. [Sadhu & Dhar, JSP, 2009]

(3) Loop erased random walk.



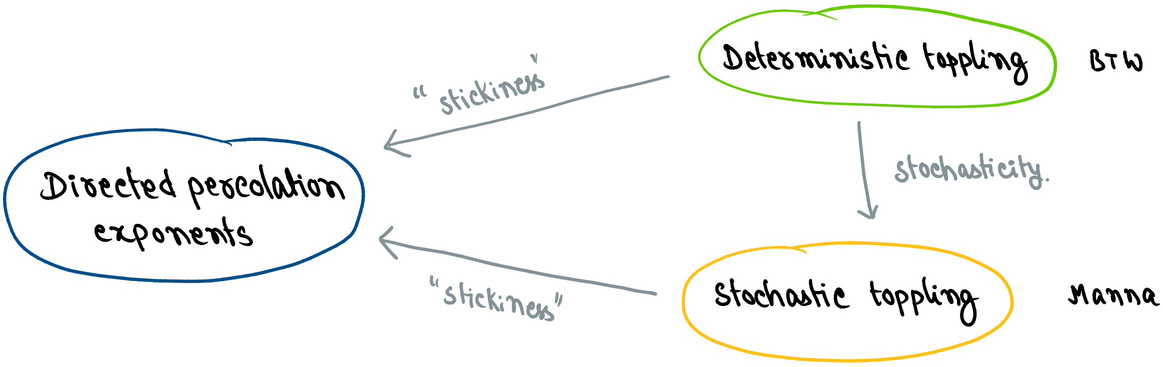
Reaches a steady state with probability of erased length of loop $P(\ell) \sim 1/\ell$ with exponential cutoff



Reaches a steady state with probability of erased length of loop $P(\ell) \sim 1/\ell^{\delta}$ with exponential cutoff.

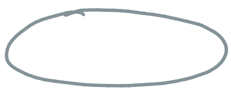
Universality in sandpile models.

There are several other models of self-organized criticality (SOC), eg, loop erased random walk, Forest fire model, mass aggregation model, etc.
Issue about their universality is not settled. Current rough picture is the following:



DP-universality describes active-absorbing phase transition with many absorbing states. Stable configs in sandpile are like absorbing states of avalanch dynamics. Infact, sandpile models are believed [Dickman et al (2000)] to be sitting at an active absorbing phase transition point tuned by the slow driving.

[Example: sandpile with periodic boundary condition (fixed energy sandpile).



No sink site ⇒ no grains are lost.

No addition of grains. Start with a fixed number of grains in a random configuration and ask whether it stabilizes or not. Numerical evidence is that there is a critical density $\rho_c < z_c - 1$ such that in the $L \to \infty$ limit, for

$\rho < \rho_c$: stabilizes with prob $\to 1$

$\rho > \rho_c$: does not stabilize with prob $\to 1$.

Ref: Anne Fey et al, PRL 104 (2010), 145703.
Dickman et al, Br. J. Phy. 30 (2000), 27.
Riddhipratim Basu et al, AHPPS 55 (2019), 1258

Then it is expected that sandpile models would belong to DP universality class.
However, additional conservation laws change universality.

Remark: Computational complexity of Abelian sandpiles.
What stable config it reaches after avalanch? Is a config recurrent?
Both of these for $d \geq 3$ is P-complete.

Ref: Moore and Nilsson, JSP 96 (1999), 205.