Graphical expansion methods
Thursday, May 20, 2021 6:06 PM

Ref: (1) lecture note by Deepak Dhar: Graphical enumeration techniques.

(2) Ch 4 of the book of Mussardo.

(3) Lecture note of David Tong.

## Graphical expansion: for Ising model.

(a) Simple example: Ising on 1d with nearest neighbor interactions with periodic boundary condition.

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$$ (zero magnetic field)

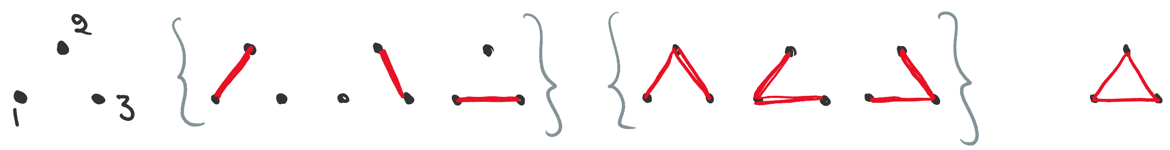
The partion function
$$Z_N = \sum_{\{\sigma_i\}} e^{+J \sum_{\langle ij \rangle} \sigma_i \sigma_j}$$

Because $\sigma_i = \pm 1$, we can write $e^{J\sigma_i\sigma_j} = (\cosh J)\left(1 + \sigma_i\sigma_j \tanh J\right)$

Then
$$Z_N = (\cosh J)^N \sum_{\sigma_1} \cdots \sum_{\sigma_N} \left(1+\sigma_1\sigma_2 v\right)\left(1+\sigma_2\sigma_3 v\right) + \cdots \left(1+\sigma_N\sigma_1 v\right)$$
$v = \tanh J$

Lets take $N=3$ (three sites)
$$Z_3 = (\cosh J)^3 \sum_{\sigma_1}\sum_{\sigma_2}\sum_{\sigma_3} \Big[1 + v\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) + v^2\left(\sigma_1\sigma_2\sigma_2\sigma_3 + \sigma_1\sigma_2\sigma_3\sigma_1 + \sigma_2\sigma_3\sigma_3\sigma_1\right) + v^3\left(\sigma_1\sigma_2\sigma_2\sigma_3\sigma_3\sigma_1\right)\Big]$$

These terms can be graphically represented as



$O(0)$ $O(v)$ $O(v^2)$ $O(v^3)$

In partion function these amplitudes are to be summed over $\{\sigma_i\}$. Because $\sigma_i = \pm 1$, $\sum_{\sigma_i} \sigma_i = 0$, and because of this graphs even number of connected edges contribute.

gives
$$Z_N = (\cosh J)^N \left\{2^N + v^N \cdot 2^N\right\} = 2^N (\cosh J)^N \left\{1 + (\tanh J)^N\right\}$$

[Same result obtained by transfer matrix]

**Remark:** for open boundary condition, only the empty graph contributes, and gives

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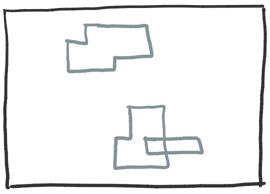
$$Z_N = 2^N (\cosh J)^{N-1}$$

This method is straightforward to generalize for other graphs, and in higher dimensions.

**Ising model on square lattice:** nearest neighbor interaction and zero field, periodic boundary.

Only connected graphs contribute.



$$Z_N = 2^N (\cosh J)^B \sum_{l=0} v^l \, n_l$$

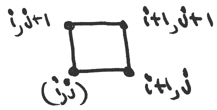
$B$ ← total number of bonds in the lattice.

$n_l$ ← numbers of connected graph of length $l$.

This way, calculation of partition function of the Ising model has been reduced to enumeration of graphs.

**Example:** the method is easy to generalize. Consider Ising model with a particular four spin coupling.

$$H = -J \sum_{i,j} \sigma_{i,j}\, \sigma_{i+1,j}\, \sigma_{i,j+1}\, \sigma_{i+1,j+1}$$

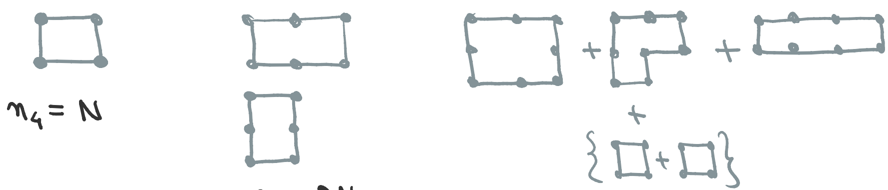



Only the lowest order graph (empty) is non zero.

$$Z_N = 2^N (\cosh J)^{2N}$$

**Back to 2d Ising model:** (Periodic boundary)

$$\frac{Z_N}{2^N (\cosh J)^{2N}} = 1 + n_4 v^4 + n_6 v^6 + n_8 v^8 + \dots$$



$n_4 = N$

$n_6 = 2N$

$$n_8 = N + 4N + 2N + \frac{N}{2}(N-5) = \frac{N}{2}(N+9)$$

$$\boxed{\Rightarrow \frac{Z_N}{2^N (\cosh J)^{2N}} = 1 + N v^4 + 2N v^6 + \frac{N}{2}(N+9) v^8 + \dots}$$

In high temperature (equivalently small $J$) $v = \tanh J$ is small. In this limit keeping few leading order terms already give a good estimate. For this reason this is called high temperature expansion.

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<u>Mayer cluster expansion</u>: a similar graphical expansion is used for interacting particles

$$Z_N = \int d\vec{r}_1 \cdots d\vec{r}_N \; e^{-\beta \sum_{i<j} U(|\vec{r}_i - \vec{r}_j|)}$$

$$= \int d\vec{r}_1 \cdots d\vec{r}_N \prod_{i<j} \left(1 + f_{ij}\right), \qquad f_{ij} \longrightarrow \left(e^{-\beta U(|\vec{r}_i - \vec{r}_j|)} - 1\right)$$

See attached reference note for more details.

<u>Free energy</u>: $-F_N = +\log Z_N$

$$= N \log\left(2[\cosh J]^2\right) + \underset{\downarrow \atop N}{n_4}\, v^4 + \underset{\downarrow \atop 2N}{n_6}\, v^6 + \underbrace{\left(n_8 - \frac{n_4^2}{2}\right)}_{\frac{N}{2}(N+9) - \frac{N^2}{2} = \frac{9}{2}N} v^8 + \cdots$$

$\Rightarrow$ free energy density

$$-f = \log\left(2[\cosh J]^2\right) + v^4 + 2v^6 + \frac{9}{2}v^8 + \cdots$$

Note, how the higher degree term $N^2$ cancells. This is expected as free energy is an extensive quantity.

<u>A note of caution</u>: In the free energy only those graphs whose degeneracy $n_\ell$ has a linear dependence on $N$ contribute to the free energy. For the example of Ising model this includes graphs which are disconnected (not linked) [for example graphs]. This is because $N$ is kept fixed.

For Meyer's cluster expansion in <u>Grand canonical</u> ensemble $N$ is not fixed, and only linked clusters contribute in the free energy. This simplification is known as the linked cluster theorem.

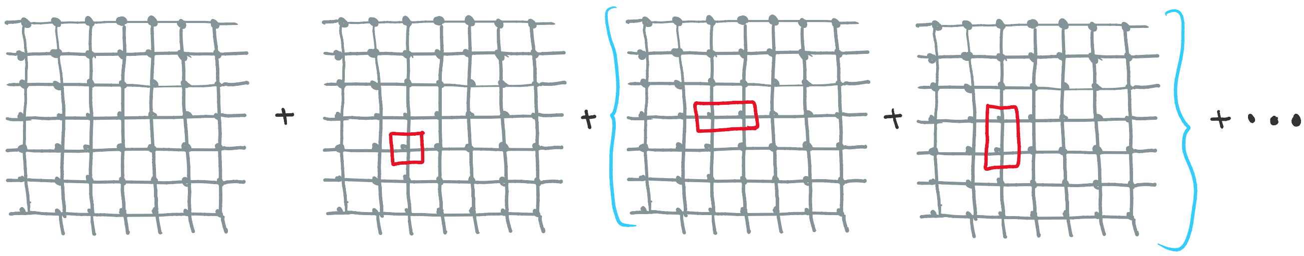
[See attached reference note on cluster expansion]

<u>Remark</u>: the graphical expansion is a powerful method in statical physics, especially for models for which exact solution is not available. Unlike other approximate methods (mean field) the errors can be made "as small as you wish" by including higher and higher order terms into account. Infact this is one of the most useful approaches for 3d Ising model where terms upto 46 th order has been calculated [PRE, 67, 066109 (2003)]. For directed percolation they have been calculated upto 171 th order. Some times it is even possible predict the full series! [For references

see the attached lecture note by Deepak Dhar].

## Low temperature expansion for 2d Ising model:

For low temperature (large $J$) excitations above the ground state are due to formation of domain wall. They can be denoted as closed loops on the dual lattice.



$$Z_N = 2\left(e^{+J}\right)^{2N}\left\{ 1 + N\left(e^{-2J}\right)^4 + 2N\left(e^{-2J}\right)^6 + \frac{N}{2}(N+9)\left(e^{-2J}\right)^8 + \cdots \right\}$$

There is a one-to-one correspondance between high temperature expansion and low temperature expansion on the dual lattice.

$$\frac{Z_N(J)}{2^N(\cosh J)^{2N}} = \frac{\tilde{Z}_N(J^*)}{2\left(e^{J}\right)^{2N}} \quad \text{with} \quad e^{-2J^*} = \tanh J \;\Rightarrow\; \boxed{\sinh 2J \sinh 2J^* = 1}$$

high temp expn — Low temp expn

**Kramers-Wannier duality:** Because square lattice is self-dual, the above correspondance between high T - low T expansion can be used to determine the transion temperature.

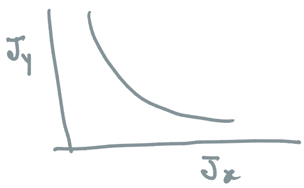
If there is a phase transion at certain value of $J$ then there is a phase transion at a related $J^*$. Assuming there is only one phase transion, it must be at the poin

$$J = J_c = J^* \Rightarrow \boxed{\sinh 2J_c = 1}$$

This is the first exact result for critical temp for 2d Ising model.

**Exercise:** For a square lattice with different horizontal and vertical coupling strength, show that there is a critical curve

$$\sinh 2J_x \sin 2J_y = 0$$




Duality is an important concept and appears in a varity of problems. In quantum theories there are duality between strong coupling and weak coupling limits. Even for out-of-equilibrium dynamics there is duality where non-equilibrium problem can be mapped to an equilibrium problem.

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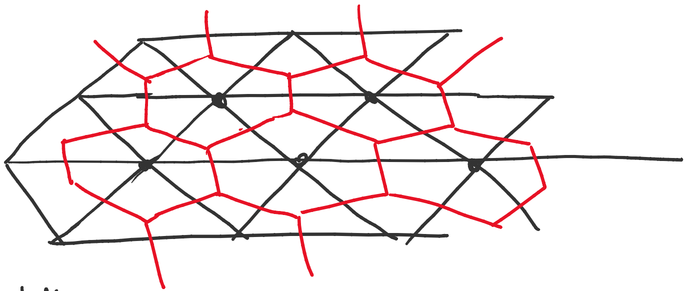
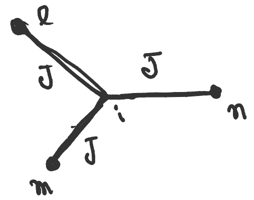
## Duality between Hexagonal and triangular lattice:



For periodic boundary, a triangular lattice of N sites is dual to a hexagonal lattice of 2N sites.

Triangular lattice

$$Z_N^{Tr} = \sum_{\{\sigma_i\}} e^{J\sigma_i\sigma_\ell + J\sigma_i\sigma_m + J\sigma_i\sigma_n}$$




high T expansion

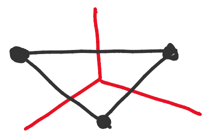
$$= 2^N (\cosh J)^{\frac{3N}{2}} \sum_\ell (\tanh J)^\ell \cdot n_\ell$$

$n_\ell$ ← number of them.

$\sum_\ell$ ↖ all closed loops on the triangular lattice.

Low temp expansion on a hexagonal lattice of 2N sites

$$Z_{2N}^{Hex} = 2(e^J)^{3N} \sum_\ell (e^{-J})^\ell \cdot n_\ell$$



↖ all closed loops on the dual triangular lattice

Then

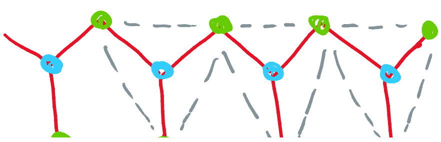
$$\frac{Z_{2N}^{Hex}(J)}{2(e^J)^{3N}} = \frac{Z_N^{tri}(J^*)}{2^N(\cosh J^*)^{\frac{3N}{2}}} \quad \text{with} \quad e^{-J} = \tanh J^*$$

$$\Rightarrow \boxed{\sinh 2J^* \sinh 2J = 1}$$

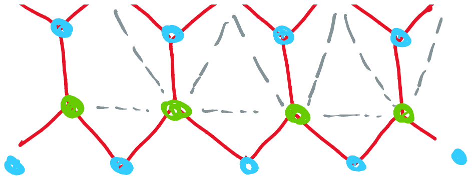
However, this duality does not immediatly help getting the critical $J_c$. This requires another relation between two lattices.

## Star-triangle Identity

This is another duality between partition function between Ising models on hexagonal and triangular lattice. First, the statement:



For a hexagonal lattice of 2N stites, with nearest neighbor coupling strength J, the partition function (upto a prefactor)



... a hexagonal lattice of 2N sites, with nearest neighbor coupling strength $J$, the partition function is same (upto a prefactor) as the partition function of the triangular lattice (made of green sites) of $N$ sites with coupling strength $K$,

$$Z_{2N}^{Hex}(J) = D^N \cdot Z_N^{tri}(K)$$

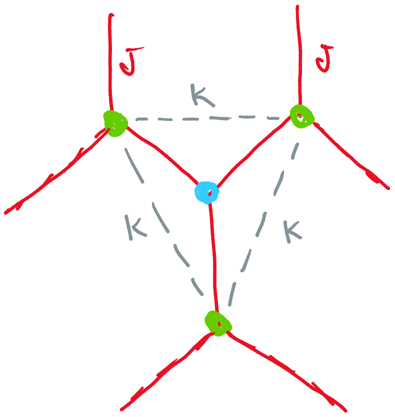
where

$$\sinh 2K \sinh 2J = h(K),$$

$$D^2 = \frac{2}{h}(\sinh 2J)^3.$$

Expression of $h(K)$ is given in ch 4.4 in the book of Mussardo.

Derivation of this relation comes using an observation that hexagonal lattice is bipartite (green and blue sites), and summing over spins on blue sites (in $Z_N$) introduces an effective coupling $K$ between the green sites.




$\Leftarrow$ this relation is called a star-triangle transformation.

For a proof, see Ch 4.4, book of Mussardo.

## Transition point

star-triangle identity gives

$$Z_{2N}^{Hex}(J) = D^N Z_N^{tri}(K)$$

High T - low T duality gives

$$Z_{2N}^{Hex}(J) = 2\left[\frac{e^{3J}}{2(\cosh J^*)^{3/2}}\right]^N \cdot Z_N^{tr}(J^*)$$

Combining the two, and after an algebra we get

$$\boxed{\begin{aligned} Z_N^{tr}(K) &= h^{\frac{N}{2}} Z_N^{tr}(J^*) \\ \text{with} \quad \sin 2J^* &= h(K)\cdot \sin 2K \end{aligned}}$$

Gives one-to-one correspondance between two values of coupling constant on same lattice.

Then, by a similar argument as before, a transition at $J^* = K = J_c \Rightarrow h(J_c) = 1$

This gives, transition point,

$$\sinh 2J_c = \begin{cases} \frac{1}{\sqrt{3}} & \text{for triangular lattice,} \\ \sqrt{3} & \text{for hexagonal lattice.} \end{cases}$$

Remark: ① $T_c^{hex} < T_c^{square} < T_c^{tri}$. This is because of higher coordination number.

② The analysis is straightforward to extend for non-isotropic interactions.