Exactly solvable models in one dimension

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Ref: Book by Giuseppe Mussardo on "statistical field theory".

① **The potts model:** One generalization of Ising model is q-state Potts model. In 1-d chain each site is assigned a variable $\sigma_i = 1, 2, \cdots, q$.

Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j} \longleftarrow \text{Kronecker delta.}$$

The Hamiltonian is invariant under permutation of the q-states. It is clear that the nature of q-values is completely inessential. They could be any numbers, colors or objects. For $q=2$, the model maps to the Ising model. A direct way of seeing this is by writing $\delta_{\sigma,\sigma'} = \frac{1}{2}(1+\sigma\sigma')$ with $\sigma = \pm 1$.

Transfer matrix: For the partition function

$$Z_N = \sum_{\{\sigma\}} e^{J\sum_{\langle ij \rangle} \sigma_i \sigma_j}$$

[Ignoring β for simplicity]

we can define transfer matrix the same way as we did for Ising model.

$$Z_N(\sigma_N) = \sum_{\sigma_{N-1}} T(\sigma_N, \sigma_{N-1})\, Z_{N-1}(\sigma_{N-1})$$

where

$$T(\sigma,\sigma') = e^{J\delta_{\sigma,\sigma'}} \equiv \begin{pmatrix} e^J & 1 & 1 & \cdots & 1 \\ 1 & e^J & & & 1 \\ 1 & & e^J & & \vdots \\ \vdots & & & \ddots & \\ 1 & 1 & \cdots & & e^J \end{pmatrix}_{q\times q}$$

The eigenvalues are

$$\lambda_+ = e^J + q - 1$$

$$\lambda_- = e^J - 1 \quad \leftarrow \text{this state } (q-1) \text{ degenerate.}$$

The largest eigenvalue $\lambda_+ > \lambda_-$. Then for periodic Potts model, the free energy density in the thermodynamic limit is

$$f = -\frac{1}{\beta}\log\left(e^J + q - 1\right)$$

Why Perron-Frobenius theorem does not apply for this transfer matrix. Because T is reducible to an upper triangular matrix by $P^{-1}TP$ using a permutation matrix P.

② **The O(n) model:** Spin in each site is an n-dimensional unit vector.

$$\vec{\sigma}_i \cdot \vec{\sigma}_i = \sum_{k=1}^{n} \left(\sigma_i^{(k)}\right)^2 = 1.$$

On a 1-d lattice, the Hamiltonian

$$H = -\sum_{i=1}^{N-1} J\, \vec{\sigma}_i \cdot \vec{\sigma}_j$$

The Hamiltonian is invariant under rotation of the vectors $\vec{\sigma}_i$ associated to the O(n) group, that's why the name. This is a model with continuous symmetry.

The partition function

$$Z_L = \int d\Omega_1\, d\Omega_2 \cdots d\Omega_L\; e^{J\sum_{\langle ij \rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j}$$

$\uparrow$ n-dimensional solid angles

$$\int d\Omega = d\theta_1 \cdots d\theta_{n-1} \left(\sin\theta_{n-2}\right)^{n-3} \left(\sin\theta_{n-1}\right)^{n-2}$$

n-dimensional solid angles

for $n\geq 2$ $\begin{cases} d\Omega = d\theta_1 \cdots d\theta_{n-1} \left(\sin\theta_{n-2}\right)^{n-3} \left(\sin\theta_{n-1}\right)^{n-2} \\ \text{with only } 0\leq\theta_1\leq 2\pi \text{ and rest } 0\leq\theta_k\leq\pi \end{cases}$

Corresponding transfer matrix

$$T(\sigma,\sigma') = e^{J\,\bar{\sigma}\cdot\bar{\sigma}'}$$

For $n=3$, the $\bar{\sigma}_i$ are 3-dimensional unit vectors and the model is known as the classical Heisenberg model. For this case the eigenvectors of the transfer matrix are easy to recognize using a wellknown expansion for 3-d unit vectors [You may have already seen this in Quantum mechanics]

$$e^{J\,\bar{\sigma}\cdot\bar{\sigma}'} = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^{\ell}\cdot \mathcal{J}_{\ell}(-iJ)\, Y^{*}_{\ell m}(\sigma)\, Y_{\ell m}(\sigma')$$

where $\mathcal{J}_\ell(x)$ is spherical Bessel function

$$\mathcal{J}_\ell(x) = -\frac{i^{\ell}}{2}\int_0^{\pi} d\theta\cdot\sin\theta\cdot e^{ix\cos\theta}\cdot P_\ell(\cos\theta)$$

ℓ^{th} order Legendre polynomial.

and $Y_{\ell m}(\bar{\sigma})$ are spherical harmonics.

Comparing the above expansion with $T=\sum_{\lambda}\lambda\,\psi^{*}_{\lambda}\psi_{\lambda}$ we recognize that the eigenvectors are $Y_{\ell m}(\bar{\sigma})$ and eigenvalues

$$\lambda_{\ell m} = 4\pi (i)^{\ell}\,\mathcal{J}_\ell(-iJ) \qquad \text{[does Not depend on } m\text{]}$$

This gives the partion function (for periodic boundary)

$$Z_L = \mathrm{Tr}\, T^{L} = \sum_{\ell=0}^{\infty} 4\pi (i)^{\ell}\,\mathcal{J}_\ell(-iJ)$$

The largest eigenvalue

$$\lambda_0 = \frac{4\pi\sinh(J)}{J}$$

[second largest $\lambda_1 = 4\pi\left(\frac{\cosh J}{J} - \frac{\sinh J}{J^2}\right)$]

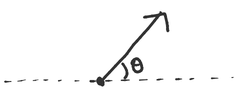
and this gives the free energy density in the theromodynamic limit

$$f = -\frac{1}{\beta}\log\left[\frac{4\pi\sinh J}{J}\right]$$

Remark: The O(n) model in 1d is exactly solvable for arbitrary n. For solution see section 2.6 of the Book by Giuseppe Mussardo. The O(2) model is the XY model.

Remark: The solution can be anlytically continued for real values of n (not only integers). This is an interesting observation, particularly $n\to 0$ limit describes dilute limit of Polymers. This is famous work of P.G. de Gennes [book: scaling concepts in polymer physics]. There is a similar mapping for Potts model, and we shall see them in graphical expansion method.

③ The Z_n model: Spins $\bar{\sigma}$ are planar unit vectors that take discrete angles



$$\theta = \frac{2\pi K}{n} \quad \text{with } K=0,1,2,\cdots,n-1.$$

corresponding Hamiltonian

$$H = -J\sum_{\langle ij\rangle}\bar{\sigma}_i\cdot\bar{\sigma}_j = -J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)$$

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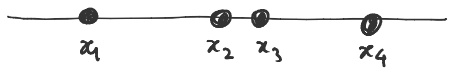
$$H = -J\sum_{\langle ij\rangle} \vec{\sigma}_i\cdot\vec{\sigma}_j = -J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)$$

This is a model with discrete symmetries. The Hamiltonian is invariant under the Abelian group $\mathbb{Z}_n$ generated by the discrete rotations.

For $n=2$, the model is the Ising model, and for $n=3$ it is equivalent to the 3 state Potts model, for $n\to\infty$ it is the $O(2)$ model.

The model is exactly solvable in 1-d.

④ Feynman gas model:



Particles in 1-d interacting by a nearest neighbour interaction potential $V(|x_i - x_{i+1}|)$

Partition function of this model can be solved exactly. [Assignment]

None of the above models show phase transition in one dimension.

Mean-field solution and mean-field models

Ref: ch 3 of the book of Mussardo.

① Weiss mean field theory of Ising model.

for a d-dimensional Ising model

$$H = -\frac{J}{2}\sum_{\langle ij\rangle}\sigma_i\sigma_j - h\sum_i \sigma_i$$

(2 is for to avoid factor 2 appearing at places)

the magnetization

$$m = \frac{1}{N}\left\langle \sum_{j=1}^{N}\sigma_j\right\rangle$$

let us write the

$$\sigma_i\sigma_j = (m+\sigma_i - m)(m+\sigma_j-m)$$
$$= -m^2 + m(\sigma_i+\sigma_j) + \underline{(\sigma_i - m)(\sigma_j-m)}$$

Mean field approximation ignores this term, whose average is the correlation of how much spins fluctuates around their mean value.

Then under this assumption

$$H = Jm^2\sum_{\langle ij\rangle} - Jm\sum_{ij}(\sigma_i+\sigma_j) - h\sum_i\sigma_i$$
$$= Jm^2\cdot\frac{N}{2}\cdot q - (Jqm+h)\sum_i\sigma_i$$

Notice how within this approximation, the interaction with other spins is replaced by an effective mean (average) field Jqm. $[\sum_{\langle ij\rangle}\sigma_i\sigma_j \to qm\sigma_i]$.

Since all spins are decoupled, partition function is easy to compute

$$Z_N = e^{-Jm^2\frac{Nq}{2}}\prod_{i=1}^{N}\sum_{\sigma_i=\pm1} e^{(Jqm+h)\sigma_i} = e^{-Jm^2\frac{Nq}{2}}\left[2\cosh(qJm+h)\right]^N$$

and free energy density

$$\boxed{f(h) = -\frac{1}{N}\log Z_N = \frac{1}{2}Jqm^2 - \log\left[2\cosh(qJm+h)\right]} \quad \text{-----(1)}$$

We still need to determine the magnetization m.

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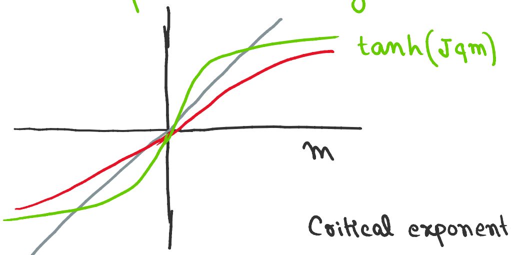
This is done using self-consistency relation

$$\boxed{m = -f'(h) = \tanh\left(Jqm + h\right)} \qquad \text{- - - - - - - - - - (2)}$$

Is there a spontaneous magnetization (for $h=0$) ?



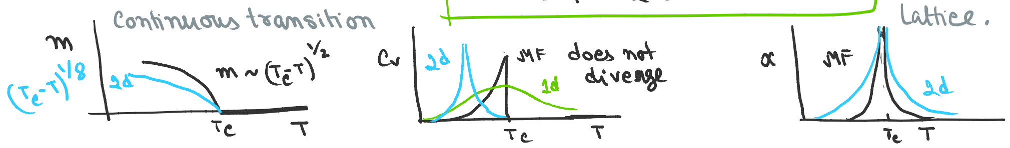
$\boxed{\text{Critical } \quad J_c = \frac{1}{q}}$

at this J_c, derivative of $\tanh(Jqm)$ at $m=0$ is 1.

Critical exponents are easy to calculate from this solution.

$$\boxed{\alpha = 0, \ \beta = \frac{1}{2}, \ \gamma = 1, \ \delta = 3}$$

Same as in Bethe Lattice.



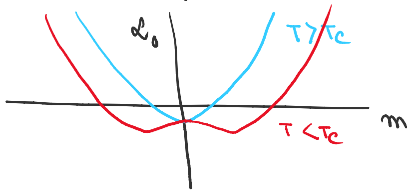
Remark: Mean field approximation is a simple way of predicting qualitative features. However there are many aspects which are unsatisfactory. For example, there is no notion of dimension left, and it predicts phase transition even in 1d $(q=2)$. Critical exponents in $d=2$ and 3 also do not give correct result. These are all because we ignored correlation of fluctuations.

Remark: The mean field solution for free energy in eq(1) and eq(2) is in a parametric form where m is a variable to be eleminated to get free energy $f(h)$.
Notice that the solution can also be written as

$$\boxed{f(h) = \min_{m} \mathcal{L}_h(m) \quad \text{with} \quad \mathcal{L}_h(m) = \frac{1}{2} Jqm^2 - \log\left[2\cosh(qJm + h)\right]}$$

This way, average magnetization corresponds to m for which $\mathcal{L}_h(m)$ is minimum.
Near transition point T_c for $h=0$, considering m being small,

$$\mathcal{L}_{h=0}(m) \sim -\log 2 + \frac{Jq}{2}\left(1 - Jq\right)m^2 + \frac{1}{12} J^4 q^4 m^4 + \dots$$



We will see this again for the Landau theory.

Mean field analysis gives the critical behavior of a model in $d \to \infty$ dimension or equivalently for models with mean field interactions.

One such model is the Ising model where each spin interacts every one else

$$J_{ij} = \frac{J}{2N} \longleftarrow \text{required for thermodynamic limit to exist.}$$

For this model, the free energy can be evaluated exactly in the thermodynamic limit [assignment] and one gets [Ref: Chowdhury & Stauffer, Ch 11.2]

$$f = \frac{1}{2} J m^2 - \log\left[2\cosh(Jm+h)\right]$$

with $\quad m = \tanh(Jm+h)$

This is the same equation obtained in Weiss mean field [substitute $q \to N$ and $J \to \frac{J}{N}$]

A second meanfield approach: variation formulation (Bragg-Williams method)

This is similar to the familiar variational approach in quantum mechanics, where for a complicated Hamiltonian that is difficult to diagonalize exactly, one can estimate the ground state energy aswell as the wavefunction by minimizing

$E = \langle \Psi | H | \Psi \rangle$ with respect to trial set of wavefunctions.

In statistical mechanics we minimize (maximize) free energy (entropy) with respect to trial probability density p. This comes from a bound where ..

free energy $\boxed{F \leq \langle H \rangle_{P_{tr}} + k_B T \langle \log P_{tr} \rangle}$, here P_{tr} is trial prob distribution.

[Proof: we have seen before (in Jarzynski equality) that for a convex function $g(x)$, $\langle g(x) \rangle \geq g(\langle x \rangle)$. A generalization gives

$$Z_L = \sum_c e^{-\beta H(c)} = \sum_c P_{tr}(c)\, e^{-\beta H(c) - \log P_{tr}(c)}$$

$$= \left\langle e^{-\beta H - \log P_{tr}} \right\rangle_{P_{tr}} \geq e^{-\beta \langle H \rangle_{P_{tr}} - \langle \log P_{tr} \rangle_{P_{tr}}}$$

$$\Rightarrow F = -\frac{1}{\beta} \log Z_L \leq \langle H \rangle_{P_{tr}} + \frac{1}{\beta} \langle \log P_{tr} \rangle_{P_{tr}} \quad]$$

The lower bound (equality) is achieved for $p(c) = \frac{e^{-\beta H(c)}}{Z}$, but determining Z is difficult. Instead we can use approximate class of $P_{tr}(c)$ and minimize within that class to get an estimate. One such class is $P_{mf}(c)$ which ignores correlation between different degrees of freedom.

An explicit example for Ising model: [Ref: Book by chowdhury and Stauffer]

For a d-dimensional Ising model with

$$H = -\sum_{ij} J_{ij} \sigma_i \sigma_j - \sum_i h_i \sigma_i$$

consider a trial probability distribution $\quad P_{tr}(\{\sigma_i\}) = \prod_i P_{m_i}(\sigma_i)$

(This is called a product measure, where correlation between different σ_i are ignored, thus mean field).

Here we introduced a parameter m_i. It is natural to consider m_i as the fluctuating magnetization at site i, over which we shall minimize. Then, we write

$$P_{m_i}(\sigma_i) = \begin{cases} \frac{1}{2}(1+m_i) & \text{for } \sigma_i = 1 \\ \frac{1}{2}(1-m_i) & \text{for } \sigma_i = -1 \end{cases}$$

[to be consistent with $\sum_{\sigma_i} p(\sigma_i) = 1$ and $\sum \sigma_i\, p(\sigma_i) = m_i$]

$$P_{m_i}(\sigma_i) = \begin{cases} \frac{1}{2}(1+m_i) & \text{for } \sigma_i = 1 \\ \frac{1}{2}(1-m_i) & \text{for } \sigma_i = -1 \end{cases}$$

[... consistent with $\sum_{\sigma_i} p(\sigma_i) = 1$ and $\sum_{\sigma_i} \sigma_i p(\sigma_i) = m_i$]

Then

$$\langle H \rangle_{P_{tr}} = -\sum_{ij} J_{ij} m_i m_j - \sum_i h_i m_i$$

$$\langle \log P_{tr} \rangle = \sum_i \left[\frac{1}{2}(1+m_i) \log \frac{1+m_i}{2} + \frac{1}{2}(1-m_i) \log \frac{1-m_i}{2} \right]$$

gives

$$\mathcal{L}_{tr}[\{m_i\}] = \langle H \rangle_{P_{tr}} + \frac{1}{\beta} \langle \log P_{tr} \rangle$$

which we minimize over $\{m_i\}$ to get bound on free energy.

Minimization condition

$$\frac{\partial \mathcal{L}_{tr}}{\partial m_i} = 0 \Rightarrow -\sum_j 2 J_{ij} m_j - h_i + \underbrace{\frac{1}{2\beta} \log \frac{1+m_i}{1-m_i}}_{\frac{1}{\beta}\operatorname{arctanh}(m_i)} = 0$$

$$\Rightarrow \boxed{m_i = \tanh\left[2\beta \sum_j J_{ij} m_j + \beta h_i \right]}$$

for nearest neighbor interaction with q neighbors, the above equation is same as we obtained in the Wiess mean field theory. (set $J_{ij} = \frac{J}{2}$ such that $2\sum_j J_{ij} = qJ$).

Remark: although $\mathcal{L}[m]$ is not same in both cases, their behavior near critical point is similar. This one can see by expandin $\mathcal{L}_{tr}$ for $h=0$ and for small m around critical point, which gives

$$\mathcal{L}(m) \simeq -\log 2 + \frac{1}{2}(1 - Jq) m^2 + \frac{1}{12} m^4 + \cdots$$

Both approaches lead to same values of critical exponents.

Bragg-Williams mean field for Potts model: [Ref: Book by Mussardo]

$$\mathcal{H} = -\sum_{ij} J_{ij} \delta_{\sigma_i \sigma_j} - \sum_i h_i \delta_{\sigma_i, 1} \qquad \text{with } \sigma_i = \{1, 2, \cdots, q\}$$

the choice to apply field in $\sigma = 1$ component is arbitrary.

follow the same procedure as for the Ising model.

$$P_{tr}[\{\sigma_i\}] = \prod_i P_{m_i}(\sigma_i) \qquad \text{(Product measure)}$$

As we chose the field to be along $\sigma = 1$ direction, our order parameter is expected to be $m_i = \langle \delta_{\sigma_i, 1} \rangle$. Then we may parametrize $P_{m_i}(\sigma_i)$ accordingly.

$$P_m(\sigma) = m\, \delta_{\sigma,1} + \frac{1-m}{q-1}(1 - \delta_{\sigma,1})$$

This is consistent with

$$\sum_\sigma P_m(\sigma) = 1 \quad \text{and} \quad \sum_\sigma \sigma P_m(\sigma) = m$$

This gives

$$\langle H\rangle_{P_{tr}} = -\sum_{ij} J_{ij} \sum_{\sigma=1}^{q} P_{m_i}(\sigma) P_{m_j}(\sigma) \quad - \sum_i h_i P_{m_i}(1)$$

$$= -\sum_{ij} J_{ij} \left\{ m_i m_j + \frac{(1-m_i)(1-m_j)}{q-1} \right\} - \sum_i h_i m_i$$

and

$$\langle \log P_{tr}\rangle = \sum_i \sum_\sigma P_{m_i}(\sigma) \log P_{m_i}(\sigma)$$

$$= \sum_i \left\{ m_i \log m_i + (1-m_i) \log \frac{1-m_i}{q-1} \right\}$$

Just as before, minimizing $\mathcal{L}[\{m_i\}] = \langle H\rangle_{P_{tr}} + \langle \log P_{tr}\rangle$ with respect to m_i gives an estimate of the free energy.

$$\frac{\partial \mathcal{L}}{\partial m_i} = 0 \Rightarrow -2\sum_{j\neq i} J_{ij} \left[m_j - \frac{(1-m_j)}{q-1} \right] - h_i + \log \frac{m_i(q-1)}{1-m_i}$$

For the simple case of $J_{ij} = \frac{1}{2} J$ for nearest neighbor sites and $h_i = 0$

$$-Jz\left(m - \frac{1-m}{q-1}\right) + \log \frac{m(q-1)}{1-m} = 0$$

$$\Rightarrow \boxed{Jz\left(m - \frac{1-m}{q-1}\right) = \log \frac{m(q-1)}{1-m}}$$

with

$$\frac{\mathcal{L}(m)}{N} \simeq -\frac{zJ}{2}\left[m^2 + \frac{(1-m)^2}{q-1}\right] \quad + m \log m + (1-m) \log \frac{1-m}{q-1}$$

We can improve on this by defining a more suitable order parameter that is zero when all q states are equally probable (ie $m = \frac{1}{q}$) and one when state $\sigma = 1$ is with prob one ($m=1$). One such choice is $m = \frac{1}{q}[1+(q-1)s]$ with order parameter $s \in [0,1]$.

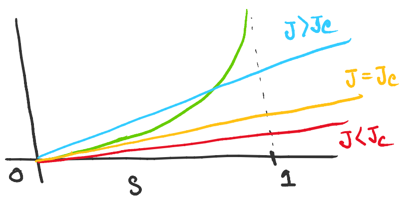
$$\boxed{\log \frac{1+(q-1)s}{1-s} = Jzs}$$

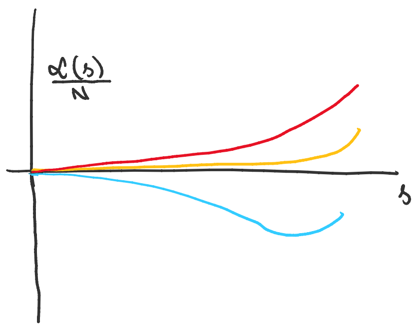
with

$$\boxed{\frac{\mathcal{L}}{N} = -\frac{zJ}{2q}\left\{1+(q-1)s^2\right\} + \frac{1+(q-1)s}{q} \log \frac{1+(q-1)s}{q} + \frac{(q-1)(1-s)}{q} \log \frac{1-s}{q}}$$

Phase transition: There are two distict scenarios.

(1) for $q<2$:

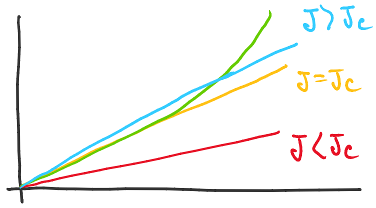


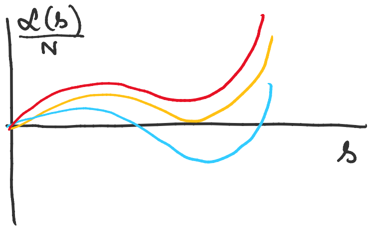


A continuous transition at $J_c = \frac{q}{z}$ where order parameter s changes smoothly from zero to a non-zero value.

A continuous transition at $J_c = \frac{q}{2}$ where order parameter s changes smoothly from zero to a non-zero value.

(2) For $q > 2$





A discontinuous transition at $J_c = \frac{2(q-1)}{z(q-2)} \log(q-1)$ where the order parameter jump from 0 to $\frac{q-2}{q-1}$.

This is also visible from the plot of $\mathcal{L}(s)$ which expanding the expression for small s gives

$$\frac{\mathcal{L}}{N} \simeq \text{const} + \frac{q-1}{2q}\left(q - zJ\right)s^2 - \frac{1}{6}(q-1)(q-2)s^3 + \dots$$

Remember this for a future discussion about Landau theory.

For more details see ch 3.2 in the book by Mussardo.

Hubbard - Stratonovich mean field theory : The weiss meanfield theory is exact for Ising model with infinite range interactions. This we can see using Hubbard-stratonovich transformation which also formulates the Ising model in terms of Euclidean Field theory.

To see this ingeneral, let's consider a general Ising Hamiltonian.

$$\mathcal{H} = -\sum_{ij} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i$$

and the corresponding partition function

$$Z = \sum_{\{\sigma_i\}} e^{\sum_{ij} J_{ij}\sigma_i\sigma_j + \sum_i h_i\sigma_i}$$

Hubbard-Stratonovich transformation introduces a set of auxilary fields ϕ_i using the gaussian integral

$$\frac{1}{\sqrt{\det(J)}}\int_{-\infty}^{\infty}\prod_i \frac{d\phi_i}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\sum_{ij}\phi_i J^{-1}_{ij}\phi_j + \sum_i \phi_i\sigma_i} = e^{\frac{1}{2}\sum_{ij}\sigma_i J_{ij}\sigma_j}$$

[You can think of this as a generalization of the gaussian integral

$$\frac{1}{\sqrt{J 2\pi}}\int_{-\infty}^{\infty} d\phi\, e^{-\frac{\phi^2}{2J} + \phi\sigma} = e^{\sigma^2\frac{J}{2}}$$

see Kardar, 2nd vol ch 3.5]

This gives the partion function

$$Z_N = \sum_{\{\sigma_i\}} \frac{1}{\sqrt{\det(2J)}}\int_{-\infty}^{\infty}\prod_i \frac{d\phi_i}{\sqrt{2\pi}}\, e^{-\frac{1}{4}\sum_{ij}\phi_i J^{-1}_{ij}\phi_j + \sum_i (h_i + \phi_i)\sigma_i}$$

[used a small modification $J \to 2J$]

$$= \frac{1}{\sqrt{\det(2J)}} \int_{-\infty}^{\infty} \prod_i \frac{d\phi_i}{\sqrt{2\pi}} \; e^{-\frac{1}{4}\sum_{ij} \phi_i J^{-1}_{ij} \phi_j} \underbrace{\prod_i 2\cosh\left[h_i + \phi_i\right]}_{2^N e^{\sum_i \log[\cosh(h_i+\phi_i)]}}$$

$$= \frac{1}{\sqrt{\det(J)}} \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{d\phi_i}{\sqrt{\pi}} \; e^{-\frac{1}{4}\sum_{ij} \phi_i J^{-1}_{ij} \phi_j + \sum_i \log[\cosh(h_i+\phi_i)]}$$

$$\Rightarrow \quad Z_N = \frac{1}{\sqrt{\det(J)}} \cdot \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{d\phi_i}{\sqrt{\pi}} \; e^{-\mathcal{L}[\{\phi_i\}]}$$

where $\mathcal{L} = \frac{1}{4}\sum_{ij} (\phi_i - h_i)\, J^{-1}_{ij}\, (\phi_j - h_j) - \sum_i \log\left[\cosh \phi_i\right]$

we make a change of variables $\phi_i \to \phi_i - h_i$

In the thermodynamic limit $N \to \infty$, $\phi_i \to \phi(\vec{r})$ and

$$Z_N \longrightarrow \int \mathcal{D}[\phi]\, e^{-\mathcal{L}[\phi]} \quad \text{with} \quad \mathcal{L} = \frac{1}{4}\int d\vec{r}\, d\vec{r}\,' \left(\phi(\vec{r}) - h(\vec{r})\right) J^{-1}(\vec{r},\vec{r}\,') \left(\phi(\vec{r}\,') - h(\vec{r}\,')\right) - \int d\vec{r}\, \log\left[\cosh \phi(\vec{r})\right]$$

Remark: Untill now we did not make any approximation. Note how the original degrees of freedom is not there any more.

To compute the path integral, the simplest approximation we can do is Saddle point approximation. [What is the large parameter? Coarse-grain scale?]

$$Z_N \simeq e^{-\mathcal{L}[\{\bar{\phi}_i\}]} \times \text{correction terms}$$

where $\bar{\phi}_i$ minimizes the Action $\mathcal{L}[\bar{\phi}_i]$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \bar{\phi}_i} = 0 \quad \text{and} \quad \frac{1}{2}\sum_j J^{-1}_{ij}(\bar{\phi}_j - h_j) = \tanh \bar{\phi}_i$$

$$\Rightarrow \quad \boxed{\bar{\phi}_i = h_i + 2\sum_j J_{ij} \tanh \bar{\phi}_j}$$

Relation to mean field approximation: In doing the Saddle point we ignored higher order terms. This approximation is equivalent to mean field approximation. To see this we need to relate the auxitary field ϕ_i to order parameter m_i.

This can be fown using relation $\quad m_i = \frac{\partial}{\partial h_i} \log Z_N$

For the result obtained by Saddle point approximation

$$m_i = -\frac{\partial \mathcal{L}[\bar\Phi]}{\partial h_i} = -\frac{1}{2}\sum_j J_{ij}^{-1}(\phi_j - h_j) = \tanh \bar\Phi_i$$

using Saddle point eq^n

$$\Rightarrow \boxed{\bar\Phi_i = \operatorname{arctanh} m_i}$$

Using this in the SPE,

$$\bar\Phi_i = h_i + 2\sum_j J_{ij} m_j$$

$$\Rightarrow \boxed{m_i = \tanh\left(h_i + 2\sum_j J_{ij} m_j\right)}$$

Same as the mean field equation for m_i.

The Gaussian model: Consider the partition function in Hubbard-Stratonovich approach (without Saddle point approximation)

$$Z_N = \frac{1}{\sqrt{\det(J)}}\int_{-\infty}^{\infty}\prod_{i=1}^{N}\frac{d\phi_i}{\sqrt{\pi}}\; e^{-\frac{1}{4}\sum_{ij}(\phi_i - h_i)J_{ij}^{-1}(\phi_j - h_j) + \sum_i \log[\cosh\phi_i]}$$

Make a change of variables $\phi_i = \sum_j 2 J_{ij}\psi_j$, gives

$$Z_N = \int\prod_{i=1}^{N}\frac{d\psi_i}{\sqrt{\pi/2}}\; e^{-\sum_{ijk\ell} J_{ik}\psi_k J_{ij}^{-1} J_{j\ell}\psi_\ell + \frac{1}{2}\sum_{ijk} J_{ik}\psi_k J_{ij}^{-1} h_j + \frac{1}{2}\sum_{ijk} h_i J_{ij}^{-1} J_{jk}\psi_k - \frac{1}{4}\sum_{ij} h_i J_{ij}^{-1} h_j + \sum_i \log\cosh(2\sum_j J_{ij}\psi_j)}$$

$$-\sum_{ik} J_{ik}\psi_k\psi_i + \sum_i h_i\psi_i - \frac{1}{4}\sum_{ij} h_i J_{ij}^{-1} h_j + \sum_i \log\cosh\left(2\sum_j J_{ij}\psi_j\right)$$

gives

$$Z_N = e^{-\frac{1}{4}\sum_{ij} h_i J_{ij}^{-1} h_j}\int\prod_{i=1}^{N}\frac{d\psi_i}{\sqrt{\pi/2}}\; e^{-\sum_{ij}\psi_i J_{ij}\psi_j + \sum_i h_i\psi_i + \sum_i \log\cosh\left(2\sum_j J_{ij}\psi_j\right)}$$

Expanding using

$$\log\cosh x = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \dots$$

leads to

$$Z_N = A\int\prod_{i=1}^{N} d\psi_i\; e^{-\sum_{ij}\psi_i J_{ij}\psi_j + \sum_i h_i\psi_i + 2\sum_i\left(\sum_j J_{ij}\psi_j\right)^2 + \dots}$$

constant independent of ψ_i

For the simplest case of nearest neighbor interactions with $J_{ij} = \frac{J}{2}$ and constant field

$$\boxed{Z_N = A\int\prod_{i=1}^{N} d\psi_i\; e^{-J\sum_{\langle ij\rangle}\psi_i\psi_j + h\sum_i\psi_i + \frac{J^2}{2}\sum_i(\psi_i)^2 + \dots}}$$

this is the Gaussian model.

Remark: the model is exactly solvable, by going to the fourier modes. [Ref. book of Mussardo, Section 7.3].

Writing $\psi(\bar{r}) = \frac{1}{\sqrt{N}} \sum_{\bar{k}} \hat{\psi}(\bar{k})\, e^{i\bar{k}\cdot\bar{r}}$ gives corresponding action

$$\mathcal{L} = A \sum_{\bar{k}} \left[m^2 + k^2\right] |\hat{\psi}(\bar{k})|^2$$

with the mass term vanishing at $J_c = \frac{1}{2}$, signaling a phase transition.

Proportional to $\propto 1 - z^2 J^2$

Below J_c (high temp phase), $m^2 > 0 \Rightarrow$ the zero mode is stable.
Above J_c (low temp phase), $m^2 < 0 \Rightarrow$ the zero mode is unstable.

Remark: Another closely related model is also known as the Gaussian model.
[Ch 3.4, book of Mussardo]

$$\mathcal{H} = -J \sum_{\langle ij\rangle} \psi_i \psi_j - h \sum_i \psi_i + \frac{1}{2}\sum_i \psi_i^2$$

where ψ_i are real continuous variables $-\infty < \psi_i < \infty$.

This model is exactly solvable in arbitrary dimension.

For $h=0$, there is a phase transition at $J_c = \frac{1}{2}$ (check this!) between a high temp phase ($J < J_c$) where all $\psi_i \sim 0$ and a low temp phase ($J > J_c$) where all spins align to a non-zero value. However this amplitud could be arbitrarily large (ψ's are not bounded) making energy arbitrarily negative. This pathological problem was corrected by Berlin & Kac by putting additional constraint

$$\sum_{i=1}^{N} \psi_i^2 = N$$

and this model is called a spherical model. [This makes the configuration space an N-dimensional spherical surface.]

Spherical model: This model is exactly solvable in arbitrary dimension. Unlike mean-field approximation solution, there are no phase transion for $d \le 2$. Moreover, the exact solution for $2 < d < 4$ shows a different behavior than mean field solution. For $d > 4$, critical behavior is similar to meanfield theory. Besides the interest due to exact solution, the model in the $n \to \infty$ limit is equivalent to the $O(n)$ model.

Exact solution in arbitrary dimension: On a d-dimensional hyper cubic lattice, with periodic boundary condition.

[Ref: book of Mussardo Section 3.5]

Partition function
$$Z_N = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\psi_i \cdot \delta\left(N - \sum_i \psi_i^2\right) \cdot e^{J\sum_{\langle ij\rangle}\psi_i\psi_j + h\sum_i \psi_i}$$

Replace it by

$$\frac{1}{\ }\int^{\infty} \ldots e^{(\mu + is)(N - \sum \psi_i^2)}$$

replace it by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} ds\, e^{(\mu+is)\left(N-\sum_i \psi_i^2\right)}$$

$$\Rightarrow \quad Z_N = \frac{1}{2\pi}\int_{-\infty}^{\infty}\prod_{i=1}^{N} d\psi_i \int_{-\infty}^{\infty} ds\; e^{-\bar{\psi}^T V \bar{\psi} + h\bar{\psi} + (\mu+is)N}$$

where in vector notation $\bar{\psi}^T = \{\psi_1, \dots, \psi_N\}$

and $V_{ij} = \begin{cases} (\mu+is) & \text{for } i=j, \\ -J/2 & \text{for } i,j \text{ nearest neighbor.} \end{cases}$

The "Action" is quadratic in ψ, and integration over ψ variables can be done using Gaussian integration [See the integration formula in Hubbard-Stratonovich transformation]. This requires an exchange in order of ψ-integration and s-integration. There is subtlety and needs μ to be large. [See book of Mussardo]

$$Z_N = \frac{1}{2}\pi^{\frac{N}{2}-1}\int_{-\infty}^{\infty} ds \cdot \frac{1}{\sqrt{\det V}}\; e^{(\mu+is)N + \frac{1}{4}\bar{h}^T V^{-1} \bar{h}} \quad \text{with} \quad \bar{h} \equiv h\cdot \mathbb{1}$$

For the periodic boundary condition on d-dimensional hyper cubic lattice L^d V is a cyclic matrix. (You need to order the index i in certain order of sites $\bar{r}$. In addition you need to use the periodic boundary condition).

Eigenvalues can be found using Fourier series method [See the book ch 3.4] and the corresponding set of eigenvalues

$$\lambda = (\mu+is) - J\left(\cos\omega_1 + \dots + \cos\omega_d\right)$$

$\underline{\underline{L^d \text{ Lattice}}}$

$$\text{With} \quad \omega = 0, \frac{2\pi}{L}, \dots, \frac{2\pi}{L}(L-1)$$

The minimum eigenvalue $\lambda_0 = \mu + is - Jd$ and the corresponding eigenvector is $\mathbb{1}$.

This allows us to explicitly calculate the partition function. For this we use

① $V\cdot\mathbb{1} = \lambda_0\mathbb{1} \Rightarrow V^{-1}\mathbb{1} = \frac{1}{\lambda_0}\mathbb{1} \Rightarrow \bar{h}^T V^{-1} \bar{h} = \frac{N h^2}{\lambda_0}$

② $\det V = \exp[\log\det V] = \exp\log\prod_n \lambda_n = \exp\sum_n \log\lambda_n$

$$= \exp\sum_{\omega_1}\dots\sum_{\omega_d}\log\left[(\mu+is) - J(\cos\omega_1 + \dots + \cos\omega_d)\right]$$

For large L ... 2π

$$\cdots \sum_{\omega_1} \cdots \sum_{\omega_d} \log\left[\cdots\right]$$

$$\xrightarrow[\text{limit}]{\text{for large } L} = \exp \frac{L^d}{(2\pi)^d} \int_0^{2\pi} \prod_i d\omega_i \, \log\left[\mu + is - J(\cos\omega_1 + \dots + \cos\omega_d)\right]$$

$$= \exp N \left\{ \log J + \underbrace{\frac{1}{(2\pi)^d} \int_0^{2\pi} \prod_i d\omega_i \, \log\left[\frac{\mu+is}{J} - \sum_{n=1}^{d} \cos\omega_n\right]}_{g\left(z = \frac{\mu+is}{J}\right)} \right\}$$

$$= J^N \exp\left(N g(z)\right)$$

Putting together

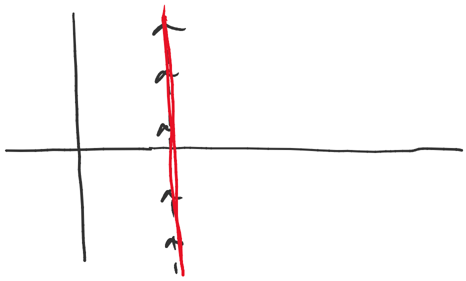
$$Z_N = \frac{1}{2\pi} \cdot \pi^{N/2} \int_{-\infty}^{\infty} ds \; J^{-\frac{N}{2}} \, e^{-\frac{N}{2} g(z)} \, e^{(\mu+is)N + \frac{Nh^2}{4\lambda_0}}$$

$z = \frac{\mu+is}{J}$; $\lambda_0 = J(z-d)$

$$= \left(\frac{\pi}{J}\right)^{\frac{N}{2}} \cdot \frac{J}{2\pi i} \cdot \int_{\frac{\mu}{J} - i\infty}^{\frac{\mu}{J} + i\infty} dz \; e^{N \phi(z)}$$

$$\phi(z) = Jz - \frac{1}{2} g(z) + \frac{h^2}{4J(z-d)}$$

The contour is taken along



As a final step,

In the thermodynamic limit $N \to \infty$, the integral can be calculated using saddle point.

For this we look at $\phi(z)$ function

$$\phi(z) = Jz - \frac{1}{2} g(z) + \frac{h^2}{4J(z-d)} \quad \text{with} \quad g(z) = \frac{1}{(2\pi)^d} \int_0^{2\pi} \prod_{i=1}^{N} d\omega_i \, \log\left[z - \sum_{i=1}^{d} \cos\omega_i\right]$$

to avoid singularity we take the contour $Re(z) > d$. This is easily done by choosing large enough μ. } Then $\phi(z)$ is analytic on the plane $Re(z) > d$.

Seeing that $\phi(z) \to \infty$ for $z = d$ and $z \to \infty$, it is expected that there a saddle point at a point z_0 on the real line which has minima along real line and maxima along imaginary line.

The saddle point is given by

$$\phi'(z_0) = 0 \Rightarrow \boxed{J - \frac{h^2}{4J(z_0-d)^2} = \frac{1}{2} g'(z_0)}$$

and then the free energy density is given by

$$\boxed{f = -\lim_{N\to\infty} \frac{\log Z_N}{N} = -\frac{1}{2}\log\left(\frac{\pi}{J}\right) + \phi(z_0)}$$

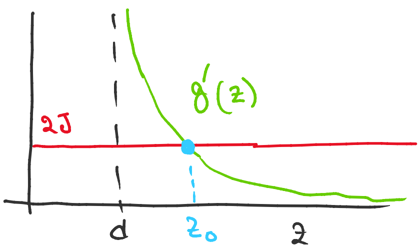
This is the solution for arbitrary $\mathbb{Z}^d$ lattice in presence of magnetic field.

Critical properties: Whether there is a phase transition or not is determined by the behavior of the saddle point z_0, particularly how it varies with control parameters (J, h). As $\phi(z)$ is an analytic function for $Re(z) > d$, any non-analytic change comes from a non analytic $z_0(J,h)$.

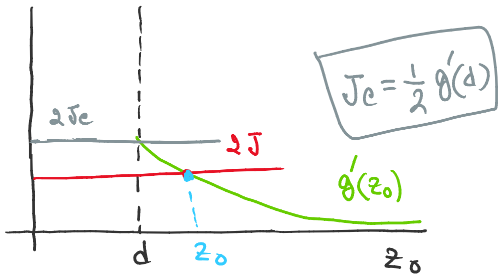
For simplicity let's put $h=0$: then, z_0 comes from solution of

$$2J = g'(z_0) = \frac{1}{(2\pi)^d}\int_0^{2\pi} \prod_{i=1}^{d} d\omega_i \frac{1}{z - \sum_{i=1}^{d}\cos\omega_i}$$

[the same function appears in Random walk on $\mathbb{Z}^d$ lattice. See assignment 2]



For $d \leq 2$



For $d > 2$

z_0 changes smoothly with J, so no phase transition.

Increasing $J > J_c$, keeps $z_0 = d$ fixed. This gives a non analytic change at J_c, therefore a phase transition. The transition is between a zero magnetization to a spontaneous magnetization state.

[See the discussion below to understand what happens at $J > J_c$, and its connection to BE condensation]

Remark: note that without the non-local constraint, there is no phase transion. (compare with the Ising model where spins are discrete)

Compared to the Gaussian model, the fixed $\sum \sigma_i^2 = N$ makes the low temp $(J > J_c)$ state stable.

Compared to the Gaussian model, the fixed $\sum \sigma_i^2 = N$ makes the low temp $(J > J_c)$ state stable.

Summary of phase behavior.

For $d \leq 2$

No phase transition

For $2 < d < 4$ and **For $d \geq 4$**

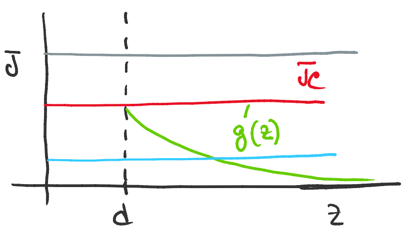
for both there is transition at $J_c = \frac{1}{2} g'(d)$

Critical exponents differ in the two regims.

For $2 < d < 4$	For $d \geq 4$
$\alpha = -\frac{4-d}{d-2}$	$\alpha = 0$
$\beta = \frac{1}{2}$	$\beta = \frac{1}{2}$
$\gamma = \frac{2}{d-2}$, $\delta = \frac{d+2}{d-2}$	$\gamma = 1$, $\delta = 3$
$\nu = \frac{1}{d-2}$, $\eta = 0$	$\nu = \frac{1}{2}$, $\eta = 0$

The phase above J_c $(T < T_c)$:

What happens to the curve



For the condition for z_0: $2J = g'(z_0)$ we see that there is an issue, because $g'(z)$ is finite at $z = d$, although we can increase J beyond J_c. The resolution is similar to that in BE condensation [see a short recap below], and related to replacing sum by an integral. The precise relation for z_0 and for $\phi(z_0)$ is with $g(z)$ replaced by

$$G(z) = \frac{1}{N} \sum_{n_1=0}^{L-1} \cdots \sum_{n_d=0}^{L-1} \log \left[z - \sum_{i=1}^{d} \cos \frac{2\pi n_i}{L} \right]$$

In our analysis we had replaced the sum by an integral. However, if we separately look at the contribution from the zeroth mode, we see that it diverges at $z = d$.

$$G(z) = \frac{1}{N} \log(z-d) + \frac{1}{N} \underbrace{\sum_{n_1} \cdots \sum_{n_d}}_{\text{except the all zero mode}} \log \left[z - \sum_{i=1}^{d} \cos \frac{2\pi n_i}{L} \right]$$

$$\simeq \frac{1}{N} \log(z-d) + g(z)$$

see that because of the $\frac{1}{N}$ term, the $\log(z-d)$ is not relevant for large N, unless for $z \to d$.

So, taking this into account, we should rewrite the free energy as (for $h=0$ case)

$$f = -\frac{1}{2} \log \frac{\pi}{J} + J z_0 - \frac{1}{2} g(z_0) - \frac{\log(z-d)}{2N} \qquad \text{for large } N$$

with z_0 from

$$2J = g'(z_0) + \frac{1}{N} \frac{1}{z_0 - d}$$

$\cdots \; N \quad z_0 - d$

Clearly, now any arbitrary J can be accomodated by the diverging term at $z_0 = d$. For $J < J_c$, $z_0 > d$ and this term is negligible for large N.

By writing
$$N = \frac{N}{2J} g'(z_0) + \frac{1}{2J(z_0 - d)} = N_1 + N_0$$
this term N_0 can be seen as the number of ψ_i in the zero mode (ground state), ie aligned.

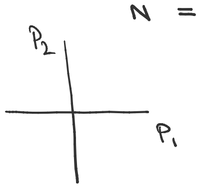
Remark: note that all these ambiguities come because we approximated $\sum$ by $\int$, for a simpler expression. This is similar to what we see in BE condensation.

BE condensation: For free bosons in 3d, in a box of volume L^3,
$$\epsilon = \frac{p_x^2 + p_y^2 + p_z^2}{2m} \quad \text{and} \quad p_i = n_i \frac{2\pi\hbar}{L} \quad \text{with } n_i = 0, \pm 1, \pm 2, \dots$$

average occupation
$$\langle n(\epsilon) \rangle = \frac{g(\epsilon)}{\frac{1}{z} e^{\beta\epsilon} - 1} \longleftarrow \text{degeneracy}$$

Note, that occupation number of $\epsilon = 0$ state can not be negative. This means
$$0 < z < 1$$

Total number of particles



$$N = \sum_{n_1} \sum_{n_2} \sum_{n_3} \frac{1}{\frac{1}{z} e^{+\beta \frac{p^2}{2m}} - 1} \simeq \frac{1}{\frac{1}{z} - 1} + \frac{L^3}{(2\pi\hbar)^3} \int_{\frac{2\pi\hbar}{L}}^{\infty} 4\pi p^2 \, dp \cdot \frac{1}{\frac{1}{z} e^{\beta \frac{p^2}{2m}} - 1}$$

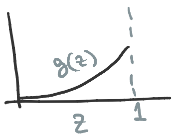
$$\simeq \frac{z}{1-z} + \frac{V \cdot 4\pi}{(2\pi\hbar)^3} \int_0^\infty dx \cdot \left(\frac{2m}{\beta}\right)^{3/2} \cdot \frac{x^2 \cdot e^{-x^2}}{\frac{1}{z} - e^{-x^2}}$$

$$\Rightarrow N = \frac{z}{1-z} + V \cdot \underbrace{\left(\frac{m}{\beta \cdot 2\pi\hbar^2}\right)^{3/2}}_{\lambda^{-3}} \cdot \underbrace{\frac{4}{\sqrt{\pi}} \int_0^\infty dx \cdot \frac{x^2 e^{-x^2}}{\frac{1}{z} - e^{-x^2}}}_{g(z)}$$

$$\Rightarrow \boxed{N = \frac{z}{1-z} + \frac{V}{\lambda^3} g(z)} \qquad \lambda = \text{de Broglie wave length.}$$

Reason for keeping the ground state term separately becomes clear now:

at $z = 1$, $g(1)$ is finite



As we can increase N arbitrarily, the way this additional particles are accounted for is from the divergence of the $\frac{z}{1-z}$ term.

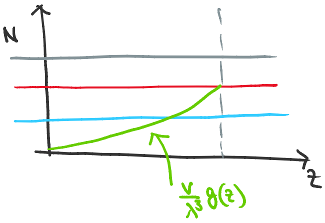
If we denote $\frac{z}{1-z} = N_0 \equiv$ occupancy at the ground level, we note that for $z < 1$, $N_0 \ll \frac{V}{\lambda^3} g(z)$ because of the large Volume V term. Only $z \to 1$, the occupancy N_0 becomes macroscopically large. This is BE condensation.

The transition: clearly transition is at

$$\boxed{N_c = \frac{V}{\lambda^3} g(1)}$$

keeping N fixed

$$T_c = \frac{2\pi\hbar^2}{k_B m \left[\frac{N}{V} g(1)\right]^{2/3}}$$



see how it vanishes for $\hbar \to 0$,
BE condensation a macroscopic manifestation of quantum nature of particles.

Remark: the ambiguity (if any) comes because we write the sum over discrete momenta as integration. This gives us a simpler function to look at, but then we have to take the ground state separately.