Second order phi4 and Feynman diagram

Tuesday, July 6, 2021 12:00 PM

Graphical representation for the $\langle V^2\rangle_0 - \langle V_0\rangle^2$ term.

Let's look at the first term

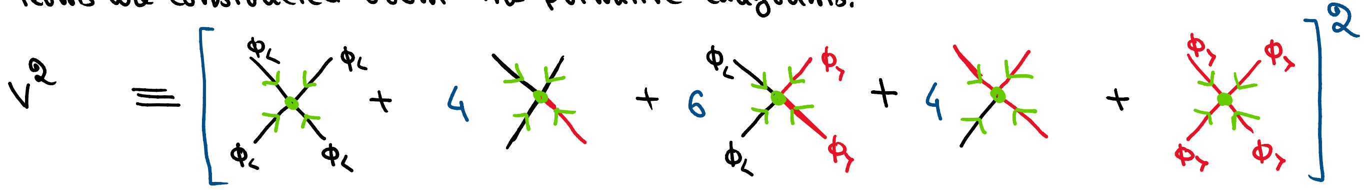
$$\langle V^2\rangle_0 = u^2 \left\langle \left[\int \frac{d\bar{q}_1\, d\bar{q}_2\, d\bar{q}_3\, d\bar{q}_4}{(2\pi)^{4d}} \phi(\bar{q}_1)\,\phi(\bar{q}_2)\,\phi(\bar{q}_3)\,\phi(\bar{q}_4)\cdot(2\pi)^d\,\delta(\bar{q}_1+\bar{q}_2+\bar{q}_3+\bar{q}_4)\right]^2 \right\rangle_0$$

We decompose the expression by writing

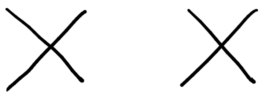
$$\int_0^{\Lambda} \frac{d\bar{q}}{(2\pi)^d}\,\phi(\bar{q}) = \int_0^{\Lambda/b} \frac{d\bar{q}_<}{(2\pi)^d}\,\phi_<(\bar{q}_<) + \int_{\Lambda/b}^{\Lambda} \frac{dq_>}{(2\pi)^d}\,\phi_>(\bar{q}_>)$$

This will generate many terms and we write them following their graphical construction.

These terms are constructed from the primitive diagrams.



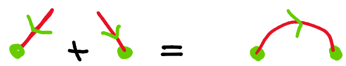
Expanding the square pairs up vertices.



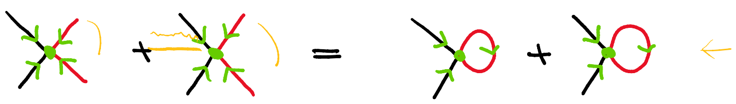
Next, the average $\langle\,\rangle_0$ using Wick's theorem connects a pair of red arms.

Each such pairing gives

$$\langle \phi_>(q)\,\phi_>(q')\rangle_0 = (2\pi)^d\,\delta(q+q')\;G(q) \Leftrightarrow$$



The $\phi_<'$s may belong to different vertex and join them. Or $\phi_<'$s may belong to the same vertex and therefore leave them disjoint. For example,



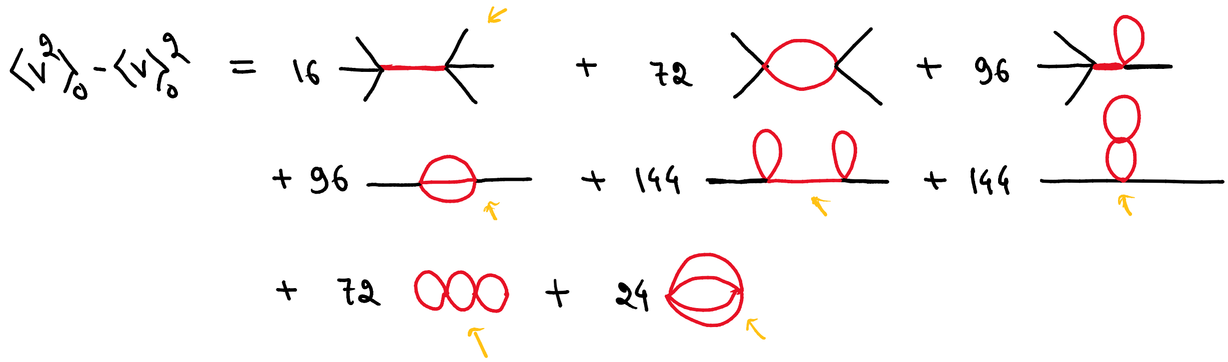
All these disjoint diagrams appear also in $\langle V\rangle_0^2$ term and therefore cancel out.

Then $\langle V^2\rangle_0 - \langle V\rangle_0^2$ only contains connected diagrams.

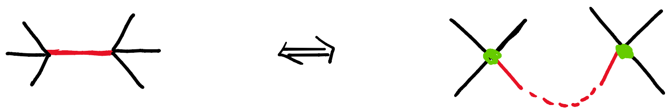
Remark: similar also extends for higher cumulant's and only connected diagrams contribute. This is generally known as linked cluster expansion.

Among these connected diagrams some have odd number of red arms which would contribute zero because average of odd number of ϕ_L is zero.

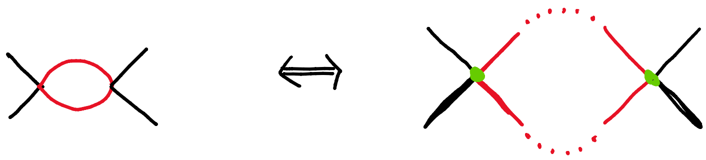
The remaining diagrams which contributes to $\langle v^2\rangle - \langle v\rangle^2$ are



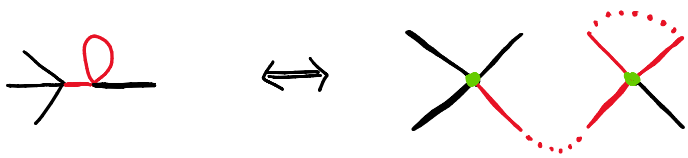
How these diagrams constructed and counting their degeneracy.



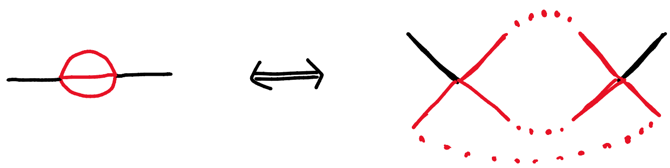
4×4



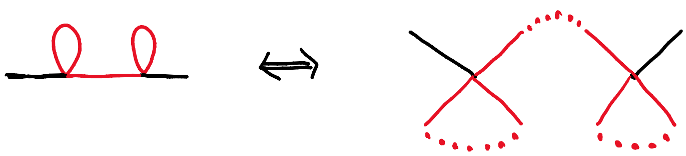
6×6×2



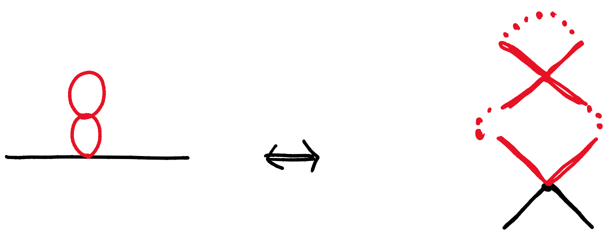
(4×4×3)×2



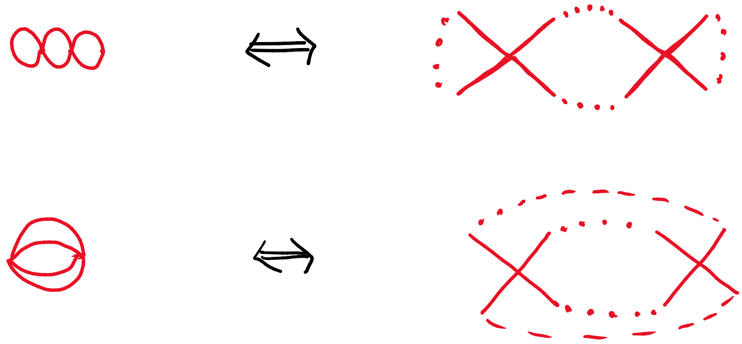
4×4×3×2



4×4×3×3

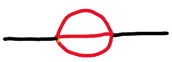
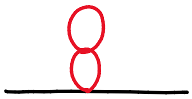


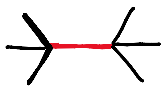
(6×1×6×2)×2

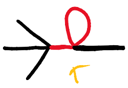
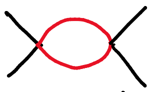


Further elimination:

(1) The last two diagrams do not have any $\phi_<$ term and thus contribute a constant term in the new $\mathcal{L}$.

(2) The diagrams , , and  contribute to the Gaussian interactions $\phi_< \phi_<$ involving powers u^2 i.e. not in the t-term. We know that around Gaussian fixed point they are irrelevant. Alternatively, you can see this as these terms are of order ϵ^2 in β-fn^c (verify by calculation) and therefore are not relevant for leading order perturbation in small ϵ. Nevertheless we keep their amplitude as a net contribution denoted by $A u^2$.

(3) Diagram  generates new interaction $\phi_<^6$. A further analysis shows that they are order ϵ^2.

(4) Diagrams  and  contribute to $\phi_<^4$ terms and they are the leading relevant terms at order ϵ.

RG equation: Performing the two additional steps (1) rescaling and (2) renormalize we get new Action

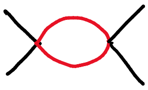
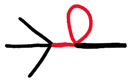
$$\mathcal{L}_{new} = \int_0^{\Lambda} \frac{d\bar{q}}{(2\pi)^d}\, r(q)\, |\phi(q)|^2 + \int \prod_{i=1}^{4} \frac{d\bar{q}_i}{(2\pi)^d} \cdot \delta\left(\sum_i \bar{q}_i\right) \cdot g(q_1, \cdots q_4)\, \phi(\bar{q}_1) \cdots \phi(\bar{q}_4)$$

with

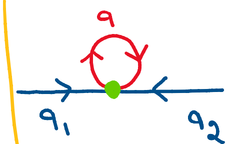
$$r(\bar{q}) = z^2 \cdot b^{-d} \cdot t + z^2 b^{-d-2} \cdot q^2 + 12$$  $$- A \cdot u^2 + \mathcal{O}(u^3)$$

with

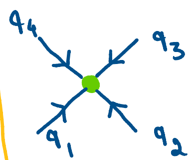
$$r(\bar q) = z^2 \cdot b^{-d} \cdot t + z^2 b^{-d-2} \cdot q^2 + 12$$  $$- A \cdot u^2 + O(u^3)$$

$$g(q_1 \cdots q_4) =$$  $$-36$$  $$-48$$  $$+ O(u^3)$$

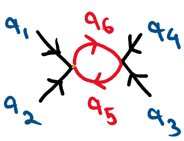
Explicit expression of the diagrams



$$= z^2 b^{-d} \cdot \int_0^{\Lambda} \frac{dq_1\, dq_2}{(2\pi)^{2d}} \phi(q_1)\, \phi(q_2) \left[\int_{\Lambda/b}^{\Lambda} \frac{dq}{(2\pi)^d} \cdot G(q) \right] u (2\pi)^d\, \delta(q_1 + q_2)$$



$$= z^4\, b^{-3d} \int_0^{\Lambda} \frac{d\bar q_1}{(2\pi)^d} \cdots \frac{d\bar q_4}{(2\pi)^d} \cdot u (2\pi)^d\, \delta(\bar q_1 + \cdots + \bar q_4) \cdot \phi(\bar q_1) \cdots \phi(\bar q_4)$$



$$= u^2 \int_0^{\Lambda/b} \frac{d\bar q_1}{(2\pi)^d} \cdots \frac{d\bar q_4}{(2\pi)^d} \cdot \phi(\bar q_1)\, \phi(\bar q_2)\, \phi(\bar q_3)\, \phi(\bar q_4)$$

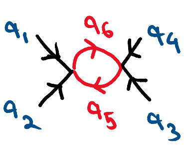
$$\int_{\Lambda/b}^{\Lambda} \frac{d\bar q_5}{(2\pi)^d} \frac{d\bar q_6}{(2\pi)^d} (2\pi)^d\, \delta(\bar q_1 + \bar q_2 + \bar q_5 - q_6) \cdot (2\pi)^d\, \delta(\bar q_3 + \bar q_4 - \bar q_5 + \bar q_6)$$

$$G(\bar q_5)\; G(\bar q_6)$$

$$= u^2 \int_0^{\Lambda/b} \frac{d\bar q_1}{(2\pi)^d} \cdots \frac{d\bar q_4}{(2\pi)^d} \cdot \phi(\bar q_1) \cdots \phi(\bar q_4)\, (2\pi)^d\, \delta(q_1 + q_2 + q_3 + q_4)$$

$$\int_{\Lambda/b}^{\Lambda} \frac{d\bar q_5}{(2\pi)^d} \cdot G(\bar q_5) \cdot G(\bar q_1 + \bar q_2 + \bar q_5)$$

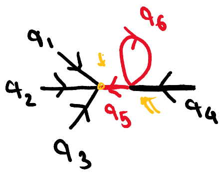
After rescaling $q \to \frac{q}{b}$ and renormalizing $\phi \to z\phi$ we get



$$= u^2 \cdot z^4 \cdot b^{-3d} \int_0^{\Lambda} \frac{d\bar q_1}{(2\pi)^d} \cdots \frac{d\bar q_4}{(2\pi)^d} \cdot \phi(\bar q_1) \cdots \phi(\bar q_4)\, \delta(\bar q_1 + \bar q_2 + \bar q_3 + \bar q_4)$$

$$\int_{\Lambda/b}^{\Lambda} \frac{dq}{(2\pi)^d} \cdot G(q) \cdot G\left(q + \frac{q_1 + q_2}{b}\right)$$

The last diagram



$$= \int_0^{\Lambda/b} \frac{d\bar q_1}{(2\pi)^d}\cdots\frac{d\bar q_4}{(2\pi)^d}\, \phi(\bar q_1)\,\phi(\bar q_2)\,\phi(\bar q_3)\,\phi(\bar q_4)$$

$$\delta(q_1+q_2+q_3+q_5)\ \delta(q_4-q_5)$$

$$\int_{\Lambda/b}^{\Lambda}\frac{d\bar q_5}{(2\pi)^d}\,G(q_5)\qquad \int_{\Lambda/b}^{\Lambda}\frac{dq_6}{(2\pi)^d}\,G(q_6)$$

$= 0$ because of the integration $\displaystyle\int_0^{\Lambda/b}\frac{dq_4}{(2\pi)^d}\cdot\int_{\Lambda/b}^{\Lambda}\frac{dq_5}{(2\pi)^d}\,\delta(q_4-q_5)$

You may also see this from the diagram itself where zero momentum condition at each vertex is not met.

Then, including this second order term and setting $z=b^{\frac{d+2}{2}}$ the RG equation becomes (the Au^2 term may contribute to the q^2 term, and this would change z. However this is $O(\epsilon^2)$ correction. See eq 5.51 – 5.53 of Kardar)

$$\boxed{\begin{aligned} t' &= b^2\left(t + 12u\,c_1 - Au^2\right) + O(u^3, u^2q^2) \quad \text{with} \quad c_1 = \int_{\Lambda/b}^{\Lambda}\frac{d\bar q}{(2\pi)^d}\cdot\frac{1}{t+q^2} \\ u' &= b^{4-d}\left[u - u^2\cdot 36\cdot c_2\right] + O(u^3, u^2q^2) \qquad c_2 = \int_{\Lambda/b}^{\Lambda}\frac{d\bar q}{(2\pi)^d}\cdot\left(\frac{1}{t+q^2}\right)^2 \end{aligned}}$$

In writing c_2 we have aproximated

$$\int_{\Lambda/b}^{\Lambda}\frac{dq}{(2\pi)^d}\cdot G(q)\cdot G\left(q+\frac{q_1+q_2}{b}\right) \sim \int_{\Lambda/b}^{\Lambda}\frac{dq}{(2\pi)^d}\cdot G(q)\cdot G(q) + \qquad q_1, q_2 +$$

This in real space means ignoring any spatial interaction in ϕ^4 term. Alternatively, expanding q_1, q_2 and converting into real integration you can see they give new terms like $\phi^2(\nabla\phi)^2$, $\phi^2\nabla^2\phi^2$, ...

A careful calculation shows that such additional terms are not relevant at $O(\epsilon)$, just like $(\nabla^2\phi)^2$ terms were not relevant for Gaussian term ϕ^2. For details see the review of Wilson & Kogut or eq (5.50) – (5.53) of Kardar.

Remark: The t'-term is important because critical point is decided by where it vanishes. However, for large Λ, $t' \sim u\Lambda^{d-2}$ is large. To get the critical point one needs to "fine tune" the original t to cancel the large Λ dependence. [David Tong lecture note]

Important: We are doing analysis at u^2 order and also ignoring terms that might contribute at ϵ^2 order or higher in β-function $(\epsilon = 4-d)$.

The RG β function

(see the calculation we did at first moment level $\langle v \rangle$)

It is convenient to introduce a parameter τ by $b = e^{d\tau}$ by treating b as small

$$\boxed{\frac{dt}{d\tau} = 2t + 12u \frac{S_d}{(2\pi)^d} \cdot \frac{\Lambda^d}{t+\Lambda^2} - A u^2 = \beta_t(t,u)} \quad \text{with} \quad S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

Similarly

$$u(\tau + d\tau) = e^{(4-d)d\tau}\left[u(\tau) - u(\tau)^2 \cdot 36 \cdot \frac{S_d}{(2\pi)^d} \cdot \frac{\Lambda^d \cdot d\tau}{(t+\Lambda^2)^2}\right]$$

$$\simeq u(\tau) + (4-d)\, d\tau \cdot u(\tau) - d\tau \cdot u(\tau)^2 \cdot \frac{36 S_d}{(2\pi)^d} \cdot \frac{\Lambda^d}{(t+\Lambda^2)^2}$$

$$\boxed{\Rightarrow \frac{du}{d\tau} = (4-d)u - Bu^2 = \beta_u(t,u) \quad \text{with} \quad B = \frac{36 S_d}{(2\pi)^d} \cdot \frac{\Lambda^d}{(t+\Lambda^2)^2}}$$

Fixed points: setting $\beta = 0$ we get two fixed points

① Gaussian fixed point: $t^* = 0$, $u^* = 0$

② WF-fixed point:

$$\left.\begin{aligned} t^* &= -\frac{6 S_d \Lambda^d}{(2\pi)^d} \cdot \frac{u^*}{t^* + \Lambda^2} + \frac{A}{2} u^{*2} \\ u^* &= \frac{(2\pi)^d}{36 S_d} \cdot \frac{(t^*+\Lambda^2)^2}{\Lambda^d} \cdot (4-d) \end{aligned}\right\} \quad \begin{aligned} t^* &\simeq -\frac{\Lambda^2}{6}\epsilon \\ u^* &\simeq \frac{(2\pi)^4}{36 S_4} \cdot \epsilon \end{aligned} \quad \text{for small } \epsilon = 4-d.$$

Remark: In writing above we made an expansion of the integrals in c_1 and c_2 in powers of $4-d = \epsilon$, treating spatial dimension as a continuous variable. Such an expansion is usually divergent, but assumed to be Borel summable. It is a very strong assumption that properties of a physical system varies smoothly with spatial dimension, which is not correct in many known cases. Nevertheless, in the problem in hand, such an expansion captures the correct

It is a very strong assumption that properties of a physical system varies smoothly with spatial dimension, which is not correct in many known cases. Nevertheless, in the problem in hand, such an expansion captures the correct observed critical behavior to good extent. (see ch 12.5.1 of Goldenfeld for expansion of these integrals)

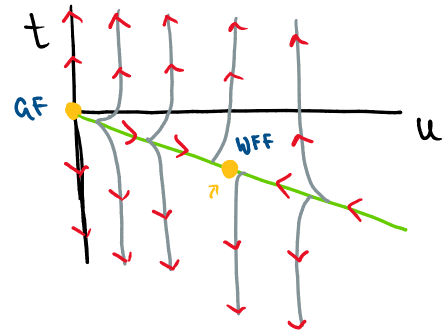
Around the WF fixed point the $\nabla\beta$-matrix

$$\begin{pmatrix} \frac{\partial \beta_t}{\partial t} & \frac{\partial \beta_t}{\partial u} \\ \frac{\partial \beta_u}{\partial t} & \frac{\partial \beta_u}{\partial u} \end{pmatrix}_{WF} = \begin{pmatrix} 2-\frac{\epsilon}{3} & \frac{3\Lambda^2}{2\pi^2}\left(1+\frac{\epsilon}{6}\right) \\ 0 & \epsilon-2\epsilon \end{pmatrix}$$

Eigenvalues give the anomalous dimensions

$$y_t \simeq 2-\tfrac{1}{3}\epsilon \quad , \quad y_2 \simeq -\epsilon \qquad \text{with} \quad \epsilon = 4-d.$$

Remarks: $y_t > 0$ but $y_2 < 0$ for $d < 4$.



WFF appears on the second eigenvector dection of the GF. For $\epsilon=0$ ie $d=4$, the WFF coincide with GF.

Eventhough the formula for t^*, u^* involve microscopic scale Λ, the y_t, y_2 are independant of it.

Remark: In this perturbation theory one can estimate values of y for other terms in $\mathcal{L}$ allowed by $\mathbb{Z}_2$ symmetry and an additial $h\phi$ term. To the linear order in ϵ, $y_h \simeq 3-\frac{\epsilon}{2}$ thus remains relevant, whereas other terms are irrelevant. (see ch 5.8 of Kardar for an estimate).

Remark: Perturbation approach does not exclude other fixed points. However, a good agreement of exponents for even $\epsilon = 1$ or 2 suggest such non-perturbative fixed point do not exist for this theory.

Critical exponents: linear order ϵ expansion gives

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$$\alpha = \frac{\epsilon}{6}, \quad \beta = \frac{1}{2} - \frac{\epsilon}{6}, \quad \gamma = 1 + \frac{\epsilon}{6}, \quad \delta = 3 + \epsilon, \quad \eta = 0, \quad \nu = \frac{1}{2} + \frac{\epsilon}{12}$$

For $\epsilon = 0$, gives the mean-field results.

Comparison for dim three by setting $\epsilon = 1$ and results from other methods.

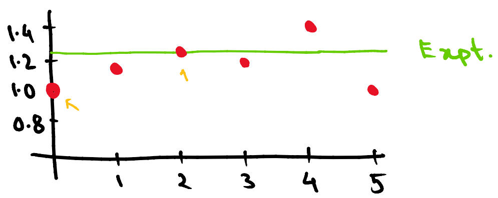
	$\epsilon = 1$	Experiment	Numerical	Mean-field.
α	0.167	0.113	0.110	0
β	0.333	0.321	0.325	0.5
γ	1.167	1.24	1.24	1
δ	4	4.75	4.82	3
ν	0.583	0.625	0.630	0.5
η	0	0.016	0.032	0

Source: Goldenfeld, table 12.1

Higher order terms in u (ie higher cumulants) is complicated to calculate. They have been derived by alternate field theoretical RG methods involving dimensional regularization etc. For example, the exponents are known to fith order in ϵ.

$$\gamma = 1 + \frac{\epsilon}{6} + 0.077\,\epsilon^2 - 0.049\,\epsilon^3 + 0.180\,\epsilon^4 - 0.415\,\epsilon^5$$

Note that terms have alternating signs.



Convergence as orders of ϵ are added.

Result improves up to ϵ^2 order, beyond which it starts to oscillate. These are characteristics of an asymptotic series.

The terms in the expansion scales as $\frac{n!}{a^n}\epsilon^n$ and as a result the expansion has zero radius of convergence. However, such expansion is Borel summable

$$\gamma = \sum_{n \geq 0} \frac{n!}{a^n}\epsilon^n = \sum_{n \geq 0} \left(\frac{\epsilon}{a}\right)^n \cdot \int_0^\infty dx \cdot x^n \cdot e^{-x}$$

$$= \int_0^\infty dx \cdot e^{-x} \sum_{n\geq 0} \left(\frac{x\epsilon}{a}\right)^n$$

$$= \int_0^\infty dx \, \frac{e^{-x}}{1 - \frac{x\epsilon}{a}} = f\left(\frac{\epsilon}{a}\right) \qquad \text{convergent for } \frac{\epsilon}{a} < 0.$$

Very good estimates for $\epsilon = 1$ $(d=3)$ and $\epsilon = 2$ $(d=2)$ by this method. However, the real power of epsilon expansion is more qualitative than quantitative. It usually, but not always, gives a reliable view of the structure of RG flows.

Remark: A similar perturbative RG can be done for $O(n)$ model for large n, by treating $1/n$ as expansion parameter.

Remark: (following David Tong lecture note) RG in QFT has a different view point. In our example Λ comes from microscopic scale on which the coarse grained theory is defined. In QFT one wishes the fields to be "fundamental", and it to be valid to arbitrarily small distance scales. One would ultimately take $\Lambda \to \infty$, and typically this leads to divergences. To avoid this problem the starting ("bare") couplings t, u as function of scale Λ where the theory is defined. Then the idea is to find $t(\Lambda)$ and $u(\Lambda)$ such that all physical quantities remain finite at $\Lambda \to \infty$. When it is possible, the theory is said to be renormalizable.