Recap

- moment generating fn^c : $\langle e^{\lambda x} \rangle$ with $\lambda \in \mathrm{Re}$
- Characteristic fun^c : $\langle e^{ikx} \rangle$
- Cumulant gen fn^c : $\log \langle e^{\lambda x} \rangle$

Q. Is moment gen. fn^c unique and analytic?

If exist for $\lambda \in [-l, l]$ for $l > 0$ then it is unique and analytic.

[Example when not unique.

take $p(x) = \dfrac{1}{x^{1+\alpha}}$

then $$\langle e^{\lambda x} \rangle = \begin{cases} 1 & \text{for } \lambda = 0 \\ \infty & \text{for } |\lambda| > 0 \end{cases}$$

for all α.]

Central limit theorem.

$M_N = \dfrac{\sum x_i}{N}$ where x_i are iid with finite $\langle x \rangle$ and $\langle x^2 \rangle_c$

Then,

$$\boxed{P\left(\frac{M_N - N\langle x \rangle}{\sqrt{N}} = z\right) \longrightarrow \frac{1}{\sqrt{2\pi \langle x^2 \rangle_c}} e^{-\frac{z^2}{2\langle x^2 \rangle_c}}}$$

$N \to \infty$

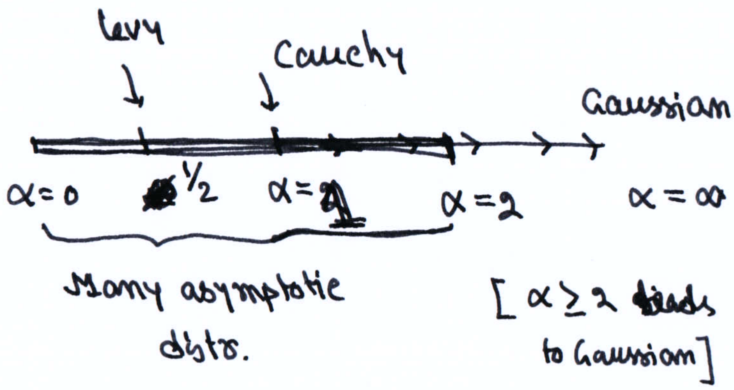
$$\Rightarrow P(M_N) \approx \frac{1}{\sqrt{2\pi N \langle x^2 \rangle_c}} e^{-\frac{(M - N\langle x \rangle)^2}{2N\langle x^2 \rangle_c}}$$

⊛ What happens for $p(x)$ where variance diverge?

Example: $p(x) \sim \frac{1}{x^{3/2}}$ for $x \gg 1$. [first passage prob]

Is there any asymptotic distribution $M = \sum_{i=1}^{N} x_i$?

Answer: There are a large number of asymptotic distributions.
(a two-parameter family)



Naive argument:

Let's redo the argument for Gaussian case.

Let $p(x)$ have finite variance. $\langle x^2\rangle_c$

Then,

← note the i.

characteristic function.

$$G_N(\lambda) = \int dM\, e^{i\lambda M} P_N(M) = \langle e^{i\lambda M}\rangle$$

$M = \sum_i x_i$

$$= \langle e^{i\lambda x}\rangle^N$$

$$\Rightarrow G_N(\lambda) = \left[g(\lambda)\right]^N \qquad g(\lambda) = \int dx\, e^{i\lambda x} p(x)$$

$$= e^{iN\lambda\langle x\rangle - N\frac{\lambda^2}{2}\langle x^2\rangle_c - iN\frac{\lambda^3}{3}\langle x^3\rangle_c + \cdots}$$

(set zero for simplicity)

Put $\lambda = \frac{K}{\sqrt{N}}$

$$\Rightarrow G_N\left(\frac{K}{\sqrt{N}}\right) = e^{-\frac{K^2}{2}\langle x^2\rangle_c + O\left(\frac{1}{\sqrt{N}}\right)} \xrightarrow[N\to\infty]{} \hat{G}(K)$$

← characteristic fn for Gaussian

⇛ This means

$$P_N(M) = \frac{1}{2\pi}\int d\lambda\, e^{-i\lambda M}\, G_N(\lambda) \qquad \text{Inverse Fourier.}$$

$$\lambda = \frac{k}{\sqrt{N}}$$

$$\simeq \frac{1}{\sqrt{N}}\cdot\frac{1}{2\pi}\int dk\, e^{-ik\cdot\frac{M}{\sqrt{N}}}\cdot \hat{G}(k) \qquad \text{large } N$$

$$= \frac{1}{\sqrt{N}}\cdot\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{M^2}{2N\sigma^2}} \qquad [\sigma^2 = \langle x^2\rangle]$$

$$\Rightarrow \boxed{P\left(\frac{M}{\sqrt{N}} = z\right) = f(z) \quad \text{Gaussian}} \qquad \text{Central limit theorem.}$$

Remark: it does not matter if $p(x)$ is symm or not. OR how it looks!

Now, what if
$$g(\lambda) \simeq e^{-c|\lambda|^{\alpha}} + \ldots$$
non-analytic at $\lambda \to 0$.

[this is what happens for $p(x) \sim \frac{1}{x^{1+\alpha}}$ with $\alpha < 2$

Example: $p(x) \approx \frac{c}{x^2} \Rightarrow g(\lambda) = e^{-c|\lambda|}$ for $\lambda \to 0$]

Repeat the excercise

$$G_N(\lambda) \simeq e^{-cN|\lambda|^{\alpha} + O(N\cdot\lambda^{1+\alpha})} \Leftarrow (g(\lambda))^N$$

$$\lambda = \frac{k}{N^{1/\alpha}}$$

$$\Rightarrow G_N\left(\frac{k}{N^{1/\alpha}}\right) = e^{-c|k|^{\alpha} + O\left(\frac{1}{N^{1/\alpha}}\right)}$$

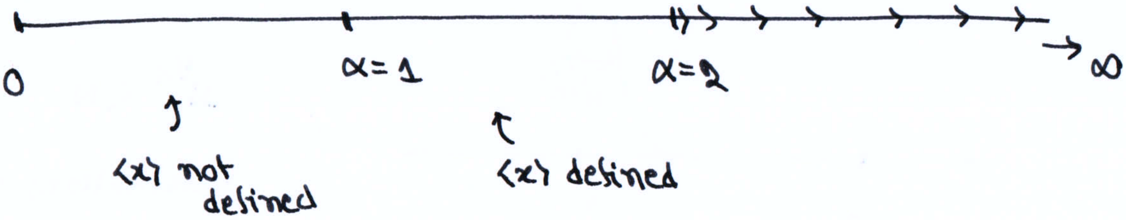
$$\Rightarrow \boxed{P\left(\frac{M}{N^{1/\alpha}} = z\right) = \mathcal{L}_{\alpha,\beta}(z)} = \frac{1}{2\pi}\int dk\, e^{-ikz}\, e^{-c|k|^{\alpha}}$$

very special case.

The general Mathematical result :

See back of Page 7.

What about $\alpha < 0$?



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flip page

$$z = \begin{cases} \dfrac{M_N - N\langle x\rangle}{N^{1/\alpha}} & \text{for} \quad 1 < \alpha \le 2, \\[2ex] \dfrac{M_N}{N^{1/\alpha}} & \text{for} \quad 0 < \alpha \le 1. \end{cases}$$

Then

$$P(z) = \mathcal{L}_{\alpha,\beta}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikz} \cdot \hat{G}_{\alpha,\beta}(k)$$

with

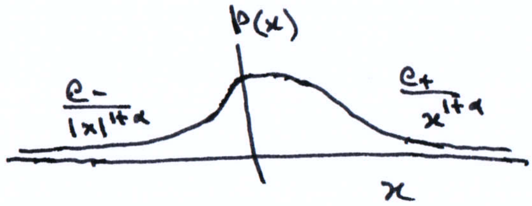
$$\hat{G}_{\alpha,\beta}(k) = e^{-|c|\,|k|^{\alpha}\{1 - i\beta\, \mathrm{sign}(k)\cdot \delta\} + ik\langle x\rangle}.$$

Here α comes from

$$p(x) \sim \begin{cases} c_-/|x|^{1+\alpha} & \text{for} \quad x \to -\infty \\[1ex] \dfrac{c_+}{|x|^{1+\alpha}} & \text{for} \quad x \to \infty \end{cases}$$

$$\beta = \frac{c_+ - c_-}{c_+ + c_-}$$

$$\delta = \begin{cases} \tan\dfrac{\pi\alpha}{2} & \text{for} \quad \alpha \ne 1 \\[1ex] -\dfrac{2}{\pi}\log|k| & \text{for} \quad \alpha = 1 \end{cases}$$



* Only tails matter !

Remark: See, unlike in CLT, how the tail decides the asymptotic distribution. This is "black swan" phenomena, where rare events have dramatic consequences.

There are no explicit formula for the asymptotic distribution $P(z)$ for arbitrary α, β. Can be written as an asymptotic series [See Kardar 1st book, eq 2.52]

Explicit results for

** we choose $\langle x \rangle = 0$ in all cases **

1. $\alpha = 2$:

$$\Rightarrow \delta = 0 \quad \Rightarrow \quad \hat{G}(k) = e^{-|ck|^2} \qquad (\beta \text{ does not matter})$$

$$\Rightarrow P(z) \text{ is Gaussian with variance } \sigma^2 = 2c^2$$

Imp Remark: note that for $p(x) \sim \frac{1}{x^3}$ variance does not exist!! Still $\sigma^2 = 2c^2$.

2. $\alpha = 1$ and $\beta = 0$ (only possible if $c_+ = c_- \Rightarrow$ symmetric $p(x)$)

$\langle z \rangle$ finite but $\langle z^2 \rangle \to \infty$

$$\hat{G}(k) = e^{-|ck|}$$

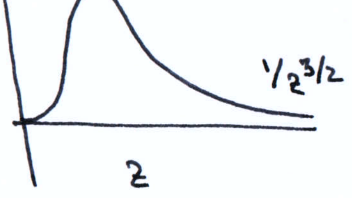
$$\Rightarrow P(z) = \frac{1}{\pi} \frac{c}{z^2 + c^2} \qquad \text{Cauchy distribution.}$$

3. $\alpha = \frac{1}{2}$ and $\beta = 1$ (true if $c_- = 0 \Rightarrow x \in [0, \infty)$)

$$\Rightarrow \delta = 1 \quad \Rightarrow \quad \hat{G}(k) = e^{-|ck|^{\frac{1}{2}}(1 + i\,\mathrm{sign}(k))}$$

even $\langle z \rangle \to \infty$.

$\Rightarrow P(z)$ is Levy distribution (* comes in first passage time)



$$P(z) = \begin{cases} \sqrt{\dfrac{c}{2\pi}} \, \dfrac{e^{-c/2z}}{z^{3/2}} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

<u>Remark:</u> One way to convince/test this asymptotic formula of $P(z)$ is by checking their stability under operation $\sum_i x_i / b(N)$

These distributions are like stable fixed points, in space of distributions.

They are also called "stable distributions".

<u>Simple examples:</u> ① Gaussian is a stable distribution.

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \longrightarrow g(\lambda) = \int dx\, e^{i\lambda x} p(x) = \langle e^{i\lambda x} \rangle$$

$$= e^{-\frac{\sigma^2}{2}\lambda^2}$$

$$\Rightarrow G(\lambda) = \langle e^{\lambda M} \rangle = [g(\lambda)]^N$$

$$= e^{-\frac{N\sigma^2}{2}\cdot\lambda^2}$$

$$\Rightarrow P(M) = \frac{1}{\sqrt{2\pi N\sigma^2}} e^{-\frac{M^2}{2N\sigma^2}}$$

Only variance changed!

<u>Leave as excercise:</u>

② Cauchy distribution.

use contour integration.

$$p(x) = \frac{1}{\pi} \frac{c}{(x-x_0)^2 + c^2} \longrightarrow g(\lambda) = e^{i\lambda x_0 - c|\lambda|}$$

$$\Rightarrow G(\lambda) = e^{i\lambda(Nx_0) - (Nc)|\lambda|}$$

$$\longrightarrow P(M) = \frac{1}{\pi} \frac{Nc}{(M - Nx_0)^2 + (Nc)^2}$$

③ Lévy distribution

$$p(x) = \sqrt{\frac{c}{2\pi}} \frac{e^{-c/2x}}{x^{3/2}} \quad \text{for } x \geq 0$$

$$\longrightarrow \quad g(\lambda) = e^{-|c\lambda|^{1/2}(1 - i\,\mathrm{sign}(\lambda))}$$

$$\Rightarrow G(\lambda) = e^{-|N^2 c \cdot \lambda|^{1/2}(1 - i\,\mathrm{sign}(\lambda))}$$

$$\Rightarrow P(M) = \sqrt{\frac{N^2 c}{2\pi}} \frac{e^{-\frac{N^2 c}{2M}}}{M^{3/2}} \quad \text{for } M \geq 0.$$

④ For general

$$g_{\alpha,\beta}(\lambda) = e^{i\lambda\langle x\rangle - |c\lambda|^{\alpha}(1 - i\beta\,\mathrm{sign}(\lambda)\cdot\delta)}$$

$$\Rightarrow G_{\alpha,\beta}(\lambda) = e^{i(N\lambda)\langle x\rangle - |(N^{\frac{1}{\alpha}}c)\lambda|^{\alpha}(1 - i\beta\,\mathrm{sign}(\lambda)\,\delta)}$$

<u>Term paper</u>: describe CLT, stable distributions using RG.

See (1) book by Sethna.

(2) An article by Jona-Lasinio.

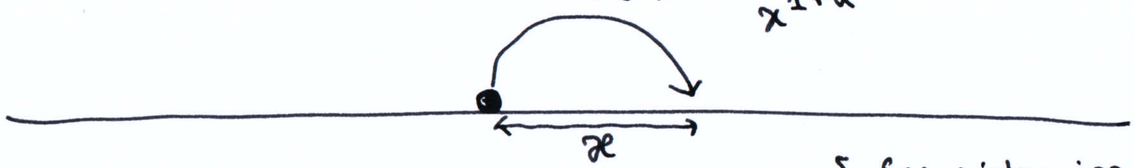
Q. Where are these ideas of limit distributions important?

In normal and anomalous diffussion in nature.

<u>Term paper:</u> "Beyond Brownian Motion" Klafter et al, Phys. today 49, 33 (1996).

<u>Simplest example:</u> Random walk (discrete time, cont space)

$$p(x) \sim \frac{1}{x^{1+\alpha}}$$



[See picture in a paper by Ariel Amir

Position after N steps

$$M_N = \sum_{i=1}^{N} x_i$$

<u>Case 1:</u> normal diffussion. $(\alpha \geq 2)$ including Gaussian.

$$\boxed{\langle M_N^2 \rangle - \langle M_N \rangle^2 \sim N} \Rightarrow P(M_N) \sim \frac{1}{\sqrt{2\pi N\sigma^2}} e^{-\frac{(M-N\langle x \rangle)^2}{2N\sigma^2}}$$

and $$P\left(\frac{M_N - \langle M_N \rangle}{\sqrt{N}} = z \right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}$$

typical displacement $\sim \sqrt{N}$

<u>Case 2:</u> anomalous diffussion $(\alpha < 2.)$

- Variance of M_N does not exist.
- $$P\left(\frac{M_N}{N^{1/\alpha}} = z \right) = \mathcal{L}_{\alpha,\beta}(z)$$

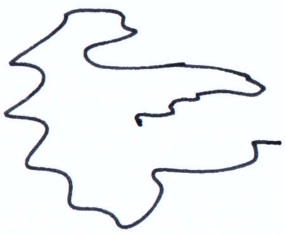
typical displacement

$$M_N \sim N^{1/\alpha} > N$$ Super-diffussion.

(for sub-diffussion see later)

Natural examples: ⓪ normal diffusion for many examples with finite correlation time. [see Sethna]

① Foraging of animals.



Flight path of an albatros fits $\boxed{\alpha = 1.}$
bumblebees & deer.

[Ref.] Viswanathan et al. Natur 381, 413 (1996).

Edwards et al. Natur 449, 1044 (2007)

② Stock prices.

$$M_N = \sum_i x_i$$

detraded price of a stock. → M_N

x_i → Price change at high temporal resolution. (~~fast~~ trading / high frequency)

$$p(x) = \cancel{\text{const}}\ \frac{c}{|x|^{1+\alpha}}\, e^{-\xi|x|}$$

with $\boxed{\alpha \simeq 1.4}$

Ref. Mantegna & Stanley, Natur 376, 46 (1995)

* A crossover happens. between Gaussian & Levy

Ref. Ismo Koponen, PRE 52, 1197 (1995)

③ Living Polymers. $\boxed{\alpha = 1.86}$

are chains of micelles that break apart & recombine to produce a stationary polymer length distribution.

One measures diffusion of a tracer micelles among other micelles.

Ref. Ott, Bouchaud, Langevin, Urbach 65, 2201, 1990.

Why super-diffusion?
Any intuition?

④ Inter stellar levy flights.

Radio waves that travel through interstellar medium (clouds of ionized gas) gets deflected randomly.

Simplest picture is that ray does a random walk due to random deflection.

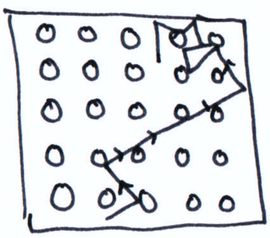
But there are claims that these are levy flights.

[Boldyrev & Gwinn, PRL 91, 131101 (2003)]

⑤ levy-flight in turbulance

[Phys Today article by Klafter "Beyond Brownian motion"]

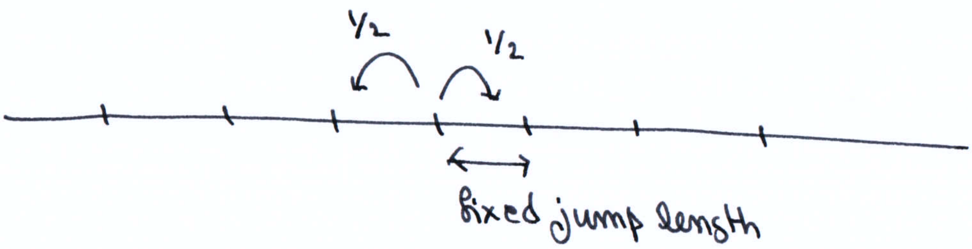
⑥ lévy flight in 2-d Lorentz gas



See papers on moodle page.

Q. Any example for ~~super~~ Sub- diffusion?

Continuous Time Random Walk (CTRW).



But waiting time chosen from $p(\tau) \sim \frac{1}{\tau^{1+\alpha}}$

In N-steps, displacement M_N

$$\Rightarrow \langle M_N^2 \rangle \sim N$$

Time in N-steps

$$t = \sum_{i=1}^{N} \tau_i$$

* note it is ~~not~~ also for $\alpha < 2$ (no variance!)

Case 1: Normal diffusion (for $1 < \alpha$)

$$t \simeq N \langle \tau \rangle + N^{1/\alpha}.* \quad \text{for large } N.$$

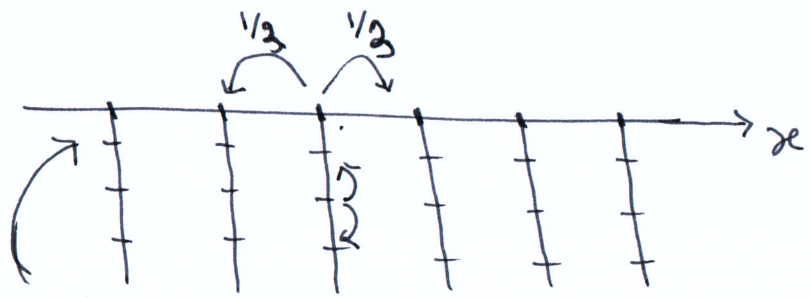
$$\Rightarrow \langle M_N^2 \rangle \simeq N \sim \frac{t}{\langle \tau \rangle} \leftarrow \text{normal diffusion.}$$

Case 2: ~~super~~ Sub-diffusion (for $\alpha \leq 1$)

$$t \simeq N^{1/\alpha} \quad \text{for large } N$$

$$\Rightarrow \langle M_N^2 \rangle \sim N \sim t^{\alpha} < t \quad \text{sub-diffusion.}$$

An example: Diffusion in Comb-lattice



time to return $\sim \frac{1}{\tau^{3/2}}$ [we will show this later. first return prob for random walk / diffusion]

$\Rightarrow$ waiting time $p(\tau) \sim \frac{1}{\tau^{3/2}} \leftarrow \alpha = \frac{1}{2}$

Gives $\langle x_t^2 \rangle \sim t^{1/2}$

Ref: diffusion in disordered media, [OR on fractals]

Havlin & Ben-Avraham

Advances in Physics 36, 695, (1987)

A combination of both worlds

$p(x) \sim \frac{1}{x^{1+\alpha_x}}$

waiting time $p(\tau) \sim \frac{1}{\tau^{1+\alpha_\tau}}$

① normal diffusion for

$\alpha_x \geq 2$ and $\alpha_\tau > 1$.

② Anomalous diffusion for else.

$X_t \sim t^{\frac{\alpha_\tau}{2\alpha_x}}$

Remark: computation of the distribution is quite difficult!

How to get Stable distributions using RG ?

Ref 1. Exercise 12.11 in Book of Sethna.

Term papers

Ref 2. "RG and probability theory", Jona-Lasinio, Phys. Rep. 352 (2001) 439.

Ref 3. "An elementary renormalization-group approach to ....". Ariel Amir, J Stat mech (2020) 013214.

Basic idea: For CLT case.

Mean $\langle y \rangle = 0$

$P_0(y)$: $y_1\ y_2\quad y_3\ y_4\quad y_5\ y_6\quad y_7\ y_8\quad y_9\ y_{10}\ \cdots$

$\downarrow$

$P_1(y)$: $y_1^{(1)}\quad y_2^{(1)}\quad y_3^{(1)}\quad y_4^{(1)}\quad y_5^{(1)}$

$$y = \frac{y_1 + y_2}{\sqrt{2}}$$

$\downarrow$

$P_2(y)$: $y_1^{(2)}\quad y_2^{(2)}\quad y_3^{(2)}$

$\vdots$

$\downarrow$

$P_\infty(y)$

$$y^{(n)} = \frac{y_1 + \cdots + y_n}{\sqrt{n}}$$

Show that

(For CLT-case)

$$P_\infty(y) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{y^2}{2\sigma^2}}$$

(take $\sigma^2 = 1$ for simplicity)

RG operation: $\quad y = \dfrac{y_1 + y_2}{\sqrt{2}}$ $\quad$ coarse-graining

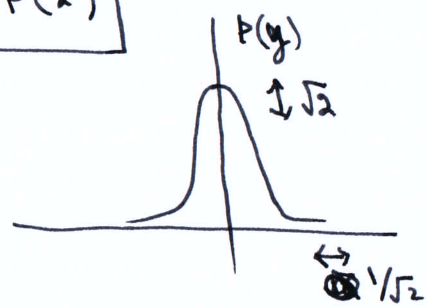
$$p_1(y) = \int dy_1\, dy_2\, p_0(y_1)\, p_0(y_2)\, \delta\left(\frac{y_1+y_2}{\sqrt{2}} - y\right)$$

$$= \sqrt{2} \int dy_1\, dy_2\, p_0(y_1)\, p_0(y_2)\, \delta(y_1 + y_2 - \sqrt{2}\, y)$$

$$= \sqrt{2} \int dy_2\, p_0(\sqrt{2}\cdot y - y_2)\, p_0(y_2)$$

$$\Rightarrow \boxed{R[p](y) = \sqrt{2} \int dx\, p(\sqrt{2}\cdot y - x)\, p(x)}$$

Rescaling



$$p_0 \xrightarrow[R]{} p_1 \xrightarrow[R]{} p_2 \cdots\cdots \xrightarrow[R]{} p_n$$

$$R^n[p](y)$$

A fixed point:

$$R[p_n^*](y) = p_{n+1}^*(y) = p_n^*$$

$$\boxed{p^*(y) = \sqrt{2} \int dx\, p^*(\sqrt{2}\, y - x)\, p^*(x)}$$

Easier to solve in Fourier-space. $\quad \hat{g}(k) = \int dy\, e^{iky}\, p(y)$

$$\Rightarrow \boxed{\hat{g}(k) = \left[\hat{g}\left(\frac{k}{\sqrt{2}}\right)\right]^2} \Rightarrow \hat{g}(k) = e^{-\frac{k^2}{2}} \text{ is a fixed point}$$

$$\Uparrow$$

$p^*(x) =$ Gaussian.

⊛ <u>generalize:</u> $y = \frac{y_1 + y_2}{b}$

$$p^*(y) = b \int dx\, p^*(x)\, p^*(by - x)$$

b is the rescaling factor.

<u>General:</u> $b = 2^{1/\alpha}$

$$\hat{g}(k) = \left[\hat{g}\left(\frac{k}{2^{1/\alpha}}\right)\right]^2$$

$$\Rightarrow \hat{g}(k) = e^{-|k|^{\alpha}}$$ is a fixed point. $\Leftrightarrow$ Symmetric Lévy distr.

[choosing b is equivalent to ~~...~~ keeping conserved quantities in RG]

[Information of b comes from your parent $p_0(x)$, knowing what is your typical fluctuations]

If be chosen incorrectly, the limiting distribution may be trivial ($\delta(x)$) or it may not exist!

Is it an attractive fixed point? [Relevant, marginal, irrelevant perturbations.]

make a small perturbation around $p^*(y)$

$$R[p^* + \epsilon\, \delta p](y) = b \int dx\, [p^* + \epsilon\, \delta p](by - x)\, (p^*(x) + \epsilon\, \delta p(x))$$

$$= p^*(y) + \epsilon [\mathcal{L} \cdot \delta p](y)$$

$$[\mathcal{L} \cdot \delta p](y) = b \int dx \left[p^*(by - x) \cdot \delta p(x) + \delta p(by - x) \cdot p^*(x)\right]$$

<u>Eigenbasis:</u>

$$[\mathcal{L} \cdot \psi_\lambda](y) = \lambda\, \psi_\lambda(y)$$

Take perturbation along any one eigen basis: $\delta p = \psi_\lambda$

$$\mathcal{R}[p^* + \epsilon\psi_\lambda] = p^* + \epsilon\lambda\psi_\lambda \Rightarrow$$

Repeated use: $\longrightarrow p^* + \epsilon\lambda^n \psi_\lambda$

$$\hat{\mathcal{R}}[g^*(k) + \epsilon\hat{\psi}_\lambda(k)] = g^*(k) + \epsilon\cdot\lambda\cdot\hat{\psi}_\lambda(k)$$

⊛ What are eigenvalues and eigenvectors: Fourier basis.

$$[\mathcal{L}\cdot\psi_\lambda](y) = \lambda\psi_\lambda(y)$$

$$\Rightarrow [\hat{\mathcal{L}}\cdot\hat{\psi}_\lambda](k) = \lambda\hat{\psi}_\lambda(k)$$

$$\boxed{2\cdot g^*\left(\tfrac{k}{b}\right)\hat{\psi}_\lambda\left(\tfrac{k}{b}\right) = \lambda\hat{\psi}_\lambda(k)}$$

$\boxed{\text{Solution}}$: $\hat{\psi}_r(k) = (ik)^r g^*(k)$ with $\lambda = \dfrac{2}{b^r}$

What it means:

$\lambda_0 = 2 > 1 \longleftrightarrow$ relevant ← but does not conserve probability.

$\lambda_1 = \dfrac{2}{b} > 1 \longleftrightarrow$ "

For $b = \sqrt{2}$

$\lambda_2 = \dfrac{2}{b^2} = 1 \longleftrightarrow$ marginal

$\lambda_3 = \dfrac{2}{b^3} < 1 \longleftrightarrow$ irrelevant.

⋮

→ We don't perturb along this direction, because it breaks conservation that $\langle x\rangle$, $\langle x^2\rangle$ are fixed!