* Langevin description: ~~[illegible]~~

A mechanical description of Brownian particle.

$$m\ddot{x} = F(x) - \frac{1}{\mu}\dot{x} + \eta(t) \qquad \text{[Newtons eq}^n\text{]}$$

with

① $\frac{1}{\mu}\dot{x}$ term is due to viscosity in the fluid medium.

For a sphere $\mu = \frac{1}{6\pi\eta R}$, R is the radius, η is viscosity.

μ is called mobility. [Ref. Kardar, vol2, ch6]

② $\eta(t)$ is a random noise due to kicks from fluid particles.

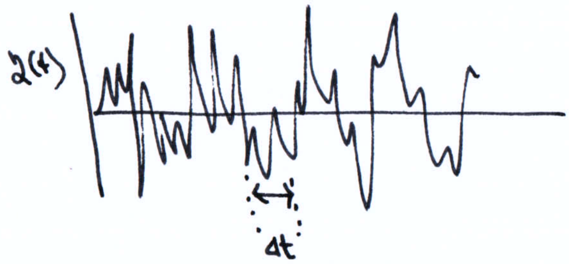
$$\left.\begin{aligned} &\langle \eta(t)\rangle = 0 \\ &\langle \eta(t)\eta(t')\rangle = 2\Gamma\,\delta(t-t') \end{aligned}\right\}$$

Gaussian noise.
All higher cumulant's vanish.

Remark: Mathematically $\eta(t)$ is not a "nice" function to deal.

A better function

$$dW_t = \int_t^{t+\Delta t} ds\, \eta(s)$$



dW_t is continuous everywhere, but nowhere differentiable.

$$P(dW_t) = \frac{1}{\sqrt{4\pi\Gamma\Delta t}}\, e^{-\frac{dW_t^2}{4\Gamma\Delta t}} \iff \langle dW_t\rangle = 0$$

$$\langle (dW_t)^2\rangle = \Delta t\cdot 2\Gamma \quad \text{(from CLT)}$$

dW_t is called a Wiener process.

Remark: Brownian particle gets energy from fluid by the $\eta(t)$ term.
and it dissipates energy back to fluid by the $\frac{1}{\mu}\dot{x}$ term.

There is a balance between the two, which gives

$$\boxed{\Gamma\cdot\mu = k_B T}$$

T = temp of the fluid.

Fluctuation dissipation relation.
[Einstein-Smoluchowski relation.]

Jean Perrin confirmed this relation in an experiment.

Gave a conclusive evidence for atomistic world.

Perrin got Nobel prize for this work.

Relation to Fokker-Planck equation:

$$\boxed{\begin{aligned} m\dot{v} &= F(x) - \frac{1}{\mu} v + \eta(t) \\ \dot{x} &= v \end{aligned}}$$

evolution of probability on (x,v)-plane is described by a F-P equation

$$\frac{\partial}{\partial t} P_t(x,v) = \mathcal{L} \cdot P_t(x,v)$$

$$= \frac{\Gamma}{m^2} \frac{\partial^2 P_t}{\partial v^2} + \frac{\partial}{\partial v}\left(\frac{\frac{1}{\mu} v - F(x)}{m} \cdot P_t \right) - v \frac{\partial}{\partial x} P_t$$

How does one show? [see next page]

Overdamped limit:

A simpler limit when inertial term $(m\dot{v})$ can be ignored. This is justified in a highly viscous fluid OR ~~for~~ low Raynold's number. This is often the case in mesoscopic length scale, for example, ~~to~~ inside biological cells.

Then,
$$m\ddot{x} = F(x) - \frac{1}{\mu}\dot{x} + \eta(t)$$

$$\longrightarrow \quad \dot{x} = \mu \cdot F(x) + \mu\, \eta(t)$$

$$= \mu F(x) + \xi(t) \qquad \text{where } \xi(t) = \mu\, \eta(t)$$

Equivalently: $\langle \xi(t) \rangle = 0$

$$\langle \xi(t)\xi(t') \rangle = 2\Gamma\mu^2 \delta(t-t')$$

Corresponding FP equation

$$\frac{\partial}{\partial t} P_t(x) = \Gamma \mu^2 \frac{\partial^2}{\partial x^2} P_t(x) - \mu \cdot \frac{\partial}{\partial x} F(x) P_t(x)$$

$$\Rightarrow \frac{1}{\mu}\frac{\partial P_t(x)}{\partial t} = (\Gamma\mu) \cdot \frac{\partial^2}{\partial x^2} P_t(x) - \frac{\partial}{\partial x}\Big(F(x) P_t(x)\Big)$$

can be absorbed by redefining time.

$k_B T$

~~usually denoted by D~~

How to go from Langevin to FP: Easy case first. Over-damped case. on infinite line.

$$\dot{x} = F(x) + \eta(t) \qquad \text{with } \langle \eta(t) \rangle = 0$$
$$\langle \eta(t)\eta(t') \rangle = 2 k_B T\, \delta(t-t')$$

~~discretise time~~

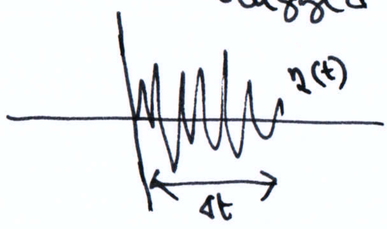
$$\Rightarrow x(t+\Delta t) = x(t) + \Delta t \Big\{ F\big(\alpha\, x(t+\Delta t) + (1-\alpha)\, x(t)\big) + dW_t$$

$\alpha \in [0,1]$ $\qquad\qquad dW_t \sim \sqrt{\Delta t}$

Remark: the α-parameter is introduced

because in the time interval Δt, the noise is rugged

then, it is not clear, at what time the force is applied!



This is a theoretical issue because there is no short-time cutoff. in calculus (stochastic).

In practical examples, the microscopic dynamics tells what α should be.

Names: $\alpha = 0$ called Itô convention.

$\alpha = \frac{1}{2}$ called Stratonovich convention.

Expanding

$$f\left(\alpha x(t+\Delta t) + (1-\alpha)\, x(t)\right)$$

$$= f\Big(x(t) + \alpha\big(x(t+\Delta t) - x(t)\big)\Big)$$

$$= \cancel{f\big(x(t) \ldots (x(t+}$$

$$= f\left(x(t) + \alpha\, dW_t + \alpha\, \Delta t \cancel{f} + \cdots\right)$$

$$= f\big(x(t)\big) + \alpha \cdot dW_t \cdot f'\big(x(t)\big) + \cdots$$

$$\Rightarrow\ x(t+\Delta t) = x(t) + dW_t + \Delta t \cdot f\big(x(t)\big) + \underbrace{\Delta t \cdot dW_t}_{\mathcal{O}(\Delta t^{3/2})} \cdot \alpha f'\big(x(t)\big) + \cdots$$

Take a test function

$$\Big\langle R\big(x(t+\Delta t)\big)\Big\rangle = \Big\langle R\big(x(t) + dW_t + \Delta t \cdot f(x(t)) + \cdots\big)\Big\rangle$$

$$= \Big\langle R(x(t)) + dW_t \cdot R'\big(x(t)\big) + \frac{dW^2}{2} R''\big(x(t)\big) + \Delta t \cdot f(x(t)) \cdot R'(x(t)) + \cdots \Big\rangle$$

$$= \big\langle R(x(t))\big\rangle + \langle dW_t\rangle \big\langle R'(x(t))\big\rangle + \frac{\langle dW_t^2\rangle}{2}\big\langle R''(x(t))\big\rangle + \Delta t \cdot \Big\langle f(x(t))\, R'\big(x(t)\big)\Big\rangle + \cdots$$

factorises because dW_t is indepent of ~~every~~ history before t.

$$\Rightarrow \Big\langle R\big(x(t+\Delta t)\big)\Big\rangle = \big\langle R(x(t))\big\rangle + \Delta t \left\{ k_B T \big\langle R''(x(t))\big\rangle + \Big\langle f(x(t))\, R'\big(x(t)\big)\Big\rangle \right\} + \cdots$$

By definition

$$\langle R(x(t)) \rangle = \int dx\, R(x)\, P_t(x)$$

$$\left\langle F(x(t))\, R'(x(t)) \right\rangle = \int dx\; F(x)\, R'(x)\, P_t(x)$$

$$= - \int dx\; R(x)\, \frac{d}{dx}\Big(F(x)\, P_t(x)\Big)$$

(* ~~Ignored boundary terms. This could be relevant~~
Boundary term is assumed to vanish. It may not happen in some cases, ~~or a~~ eg reflecting boundary)

$$\langle R''(x(t)) \rangle = \int dx\; R(x) \cdot P_t''(x)$$

Putting together

$$\int dx\; R(x) \left\{ \frac{P_{t+\Delta t}(x) - P_t(x)}{\Delta t} + \Big(F(x)\, P_t(x)\Big)' - k_B T\, P_t''(x) \right\} = 0$$

is an arbitrary test fnc. $\Rightarrow$

$$\boxed{\frac{dP_t(x)}{dt} = - \frac{d}{dx}\Big(F(x)\, P_t(x)\Big) + k_B T\, \frac{d^2}{dx^2} P_t(x)}$$

no α- dependence!

<u>Remark</u>: <u>α- does not play a role for additive noise.</u> * * *

It is important for multiplicative noise (eg. $T(x)$ space dependent)

<u>Exercise</u>: Derive the F-P equation for

$$m\ddot{x} = F(x) - \frac{1}{\mu}\dot{x} + \eta(t).$$

More about α: Itô vs Stratonovich.

α - plays a role for multiplicative noise.

For example

$$\dot{x} = f(x) + \eta(t) \quad \text{with} \quad \langle \eta(t) \rangle = 0$$

$$\langle \eta(t') \eta(t) \rangle = 2 k_B T(x)\, \delta(t'-t)$$

Exercise:

Show: corresponding FP-equation

$$\frac{d}{dt} P_t(x) = -\frac{d}{dx}\left(f(x) P_t(x)\right) + \frac{d^2}{dx^2}\left(k_B T(x) P_t(x)\right)$$

$$- \alpha \frac{d}{dx}\left(k_B T'(x) P_t(x)\right)$$

α matters!

Remark: ~~α can be determined from~~

α can be chosen either

① from the microscopic dynamics

② from physical arguments.

Example: $m\dot{v} = -\gamma v + \eta$ with $\langle \eta(t') \eta(t) \rangle = 2 k_B T \gamma\, \delta(t'-t)$

This is additive noise. Corresponding FP equation

$$\frac{dP(v)}{dt} = \frac{d}{dv}\left\{ \frac{\gamma v}{m} P + \frac{\gamma k_B T}{m^2} \cdot P' \right\}$$

Gives stationary solution

$$P_{st}(v) = \sqrt{\frac{m}{2\pi kT}}\; e^{-\frac{mv^2}{2kT}}$$

This means, for energy $E = \frac{1}{2} m v^2$

$P(v)\,dv = P(E)\,dE \longrightarrow$ $$P_{st}(E) = \frac{1}{\sqrt{\pi k T E}}\, e^{-E/kT}$$

On the otherhand, from the v-Langevin equation we get

$$\dot{E} = -\frac{2\gamma}{m} E + \eta(t)$$

with $\langle \eta(t')\eta(t)\rangle = \frac{4k_BT\cdot\gamma\cdot E}{m}\,\delta(t'-t)$

this is multiplicative noise.

Corresponding F-P equation

$$\frac{dP(E)}{dt} = \frac{d}{dE}\left(\frac{2\gamma}{m}E\cdot P\right) + \frac{4k_BT\gamma}{m}\frac{d}{dE}\left(E\frac{dP}{dE}\right)$$

$$-\alpha\cdot\frac{4k_BT\gamma}{m}\cdot\frac{dP}{dE}$$

~~The $P_{st}(E)$ obtained earlier~~

The stationary distribution $P_{st}(E)$ obtained from the FP eq^n matches with earlier $P_{st}(E)$ only for $\alpha = \frac{1}{2}$. Therefore one must choose Stratonovich convention.

Example: there is similar example for reaction-diffusion model which gives Itô as natural choice.

See assignment problem.

Example: If the $\delta(t'-t)$ in the noise correlation is considered as limit of $\lim_{\epsilon\to 0}\frac{1}{\sqrt{2\pi\epsilon}}e^{-\frac{(t'-t)^2}{2\epsilon}}$, then Stratonovich is the consistent choice.

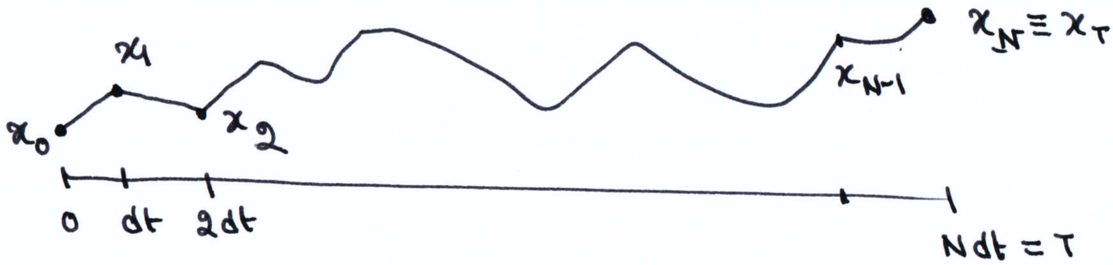
Path integral for Langevin equation

Take the simplest example

$$\dot{x} = F(x) + \eta(t) \qquad \text{with} \quad \langle \eta(t) \rangle = 0$$

Q. What is prob of a history $x(t)$?

$$\langle \eta(t') \eta(t) \rangle = 2D\, \delta(t'-t)$$



discretize

$$x_{n+1} - x_n = dt\, F\big(\alpha x_{n+1} + (1-\alpha) x_n\big) + (W_{n+1} - W_n)$$

here we wrote .... (1)

$$dW_t \equiv W_{n+1} - W_n \qquad \text{with} \quad t = n\,dt.$$

$$P(dW_t) = \frac{1}{\sqrt{4\pi D\, dt}}\, e^{-\frac{dW_t^2}{4D\,dt}} \Longrightarrow P(W_{n+1} - W_n) = \frac{1}{\sqrt{4\pi D\, dt}} \cdot e^{-\frac{(W_{n+1}-W_n)^2}{4D\,dt}}$$

Say, we want the prob of a path

$$x(t) \equiv \{x_0, x_1, \cdots\cdots, x_N\} \qquad \text{with fixed } x_0 \text{ and } x_N \equiv x_T$$

This is generated by

$$W(t) \equiv \{W_0, W_1, \cdots\cdots, W_N\} \qquad \text{with } W_0 = 0$$

and W_N fixed by condition (1)

$$\mathrm{Prob}(\omega_1, \dots, \omega_{N-1}) = \left(\frac{1}{4\pi D\,dt}\right)^{\frac{N}{2}} e^{-\frac{1}{4D\,dt}\sum_{n=0}^{N-1}(\omega_{n+1}-\omega_n)^2}$$

with ω_N fixed by ① and $x_N \equiv x_T$.

To convert into probability of x's we use

$$\mathrm{Prob}(\omega_1, \dots, \omega_{N-1})\, d\omega_1 \cdots d\omega_{N-1} = \mathrm{Pro}(x_1, \dots, x_{N-1})\, dx_1 \cdots dx_{N-1}$$

gives

$$\mathrm{Pro}(x_1 \dots x_{N-1}) = \mathrm{Prob}(\omega_1 \dots \omega_{N-1}) \cdot \underbrace{\mathrm{Det}\left[\frac{d\omega_i}{dx_j}\right]}_{\text{Jacobian.}}$$

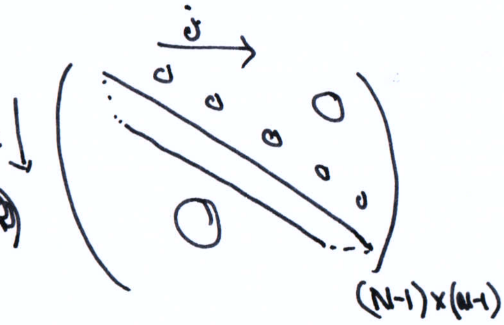
Writing eq (1) as

$$\omega_i = \omega_{i-1} - dt \cdot F\left(\alpha x_i + (1-\alpha)x_{i-1}\right) + x_i - x_{i-1}$$

We see that $\left[\frac{d\omega_i}{dx_j}\right]$ is non zero for $i=j$ and $i=j+1$,

therefore it is a lower-triangular matrix.

Corresponding determinant



$$\mathrm{Det}\left[\frac{d\omega_i}{dx_j}\right] = \prod_{i=1}^{N-1} \frac{d\omega_i}{dx_i}$$

$$= \prod_{i=1}^{N-1}\left(1 - dt\cdot\alpha\cdot F'\left(\alpha x_i + (1-\alpha)x_{i-1}\right)\right)$$

$$\simeq e^{-dt\cdot\alpha\cdot\sum_{i=1}^{N-1} F'\left(\alpha x_i + (1-\alpha)x_{i-1}\right)}$$ for small dt.

Gives

$$\text{Prob}\left(x_1 \cdots x_{N-1}\right) = \left(\frac{1}{4\pi D\, dt}\right)^{\frac{N}{2}} \cdot e^{-S}$$

with

$$S = \frac{dt}{4D} \sum_{n=0}^{N-1} \left\{ \frac{x_{n+1}-x_n}{dt} - F\left(\alpha x_{n+1} + (1-\alpha)x_n\right) \right\}^2$$

$$+ \alpha \cdot dt \sum_{n=0}^{N-2} F'\left(\alpha x_{n+1} + (1-\alpha) x_n\right)$$

In the $dt \to 0$ limit and taking $N \to \infty$ we denote

$$\text{Prob}\left(x_1 \cdots x_{N-1}\right) dx_1 \cdots dx_{N-1} = e^{-S} \cdot \mathcal{D}[x(t)]$$

with $$S = \int_0^T dt \left\{ \frac{\left(\dot{x} - F(x)\right)^2}{4D} + \alpha F'(x) \right\} = \int_0^T dt\, \mathcal{L}(x, \dot{x})$$

note, the dependence on α.

In summary

$$\text{Prob}\left(x_T, T \mid x_0, 0\right) \equiv G\left(x_T, T \mid x_0, 0\right) \leftarrow$$ called propagator / Green's function

$$= \int_{x_0}^{x_T} \mathcal{D}[x]\, e^{-S[x]}$$

$$= \lim_{\substack{dt \to 0 \\ N \to \infty}} \left(\frac{1}{4\pi D\, dt}\right)^{\frac{N}{2}} \cdot \int \prod_{n=1}^{N-1} dx_n \cdot e^{-\alpha \cdot dt \cdot \sum_{n=0}^{N-2} F'\left(\alpha x_{n+1} + (1-\alpha) x_n\right)}$$

$$\cdot e^{-\frac{dt}{4D} \sum_{n=0}^{N-1} \left\{ \frac{x_{n+1}-x_n}{dt} - F\left(\alpha x_{n+1} + (1-\alpha) x_n\right) \right\}^2}$$

Summary of what we did in a short hand notation.

$$\dot{x} = F(x) + \eta(t)$$

$$G(x_T,T|x_0,0) = \int \mathcal{D}[\eta]\, e^{-\int_0^T dt \frac{\eta(t)^2}{4D}}$$

$$= \int_{x_0}^{x_T} \mathcal{D}[x] \cdot \mathrm{Det}\left(\frac{\delta[\eta]}{D[x]}\right) \cdot e^{-\int_0^T dt \cdot \frac{(\dot{x}-F)^2}{4D}}$$

$$= \mathrm{Det}\left(\frac{\delta \eta(t)}{\delta x(t')}\right)$$

$$= \mathrm{Det}\left[\left(\frac{\partial}{\partial t} - F(x(t))\right)\delta(t-t')\right] \equiv \mathrm{Det}\left(\partial_t \pm F'(x)\right) = e^{-\alpha \int_0^T dt\, F'(x)}.$$

Remark (for people with interest in field theory. Bit technical)

The Jacobian term plays a role in Supersymmetric formulation of the Langevin equation.

Ref: Parisi and Sourlas, Phys. rev. lett, 43 (1979), 744.

They expressed the Det in terms of Grassman fields

$$\boxed{\mathrm{Det}\left(+\partial_t \pm F'(x)\right) = \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\, e^{-\int dt\, \bar{\psi}[+\partial_t \pm F']\psi}}$$

[in QFT they are Ghost fields]

Then, the propagator

$$G(x_T,T|x_0,0) = \int \mathcal{D}[x,\psi,\bar{\psi}]\, e^{-S_{PS}}$$

with $$S_{PS} = \int dt \left\{ \frac{(\dot{x}-F(x))^2}{4D} + \bar{\psi}[\partial_t \pm F']\psi \right\}$$

This action can be shown to be supersymmetric meaning invariant under transformations which mix Fermionic and Bosonic operators.

[Ref: lecture note of Kay Wiese, on statistical field theory.]

Remark: Another way the determinant can be computed is by using

$$\det M = e^{\operatorname{tr} \log M}$$

Ref: Book of Zinn-Justin on QFT.

Remark: We showed, the Action

$$S = \frac{1}{4D}\int_0^T dt\,(\dot{x}-F)^2 + \alpha\int_0^T dt \cdot F'(x)$$

It seems to depend on α, but we showed that for additive noise, the FP-equation do not depend on α, so the Action should not!

Resolution of this comes from the subtle difference of usual calculus with stochastic calculus.

To see this, let us recall that path integral is defined by time-discretization. Let's look at the term in discretized Action

$$\frac{dt}{4D}\left[\frac{x_{n+1}-x_n}{dt} - F(\bar{x})\right]^2 = \frac{(x_{n+1}-x_n)^2}{4D\,dt} - \frac{1}{2D}(x_{n+1}-x_n)F(\bar{x}) + \frac{dt}{4D}F(\bar{x})^2$$

Here

$$F(\bar{x}) = F\big(\alpha x_{n+1} + (1-\alpha)x_n\big)$$

$$= F(x_n) + \alpha(x_{n+1}-x_n)\cdot F'(x_n) + \dots$$

Recall

$$x_{n+1}-x_n = \underbrace{(W_{n+1}-W_n)}_{\mathcal{O}(\sqrt{dt})} + dt\cdot F$$

see eq (1)

gives,

$$F(\bar{x})^2 = F(x_n)^2 + \mathcal{O}(\sqrt{dt})$$

and

$$\frac{1}{2D}(x_{n+1}-x_n)F(\bar{x}) = \frac{(x_{n+1}-x_n)}{2D}F(x_n) + \frac{\alpha}{2D}(x_{n+1}-x_n)^2\cdot F'(x_n) + \dots$$

$$= \frac{(x_{n+1}-x_n)}{2D}F(x_n) + \frac{(W_{n+1}-W_n)^2}{2D}\cdot\alpha\cdot F'(x_n) + \dots$$

Recall,

$$\langle W_{n+1}-W_n\rangle = \langle dW_t\rangle = 0$$

$$\langle (W_{n+1}-W_n)^2\rangle = \langle dW_t^2\rangle = 2D\,dt$$

$$\Rightarrow \boxed{W_{n+1}-W_n \equiv dW_t = \sqrt{2D\,dt} + \text{higher order in } dt.}$$

Important!

This gives

$$\frac{1}{2D}(x_{n+1}-x_n)F(\bar{x}) = \frac{1}{2D}(x_{n+1}-x_n)F(x_n) + dt\,\alpha\cdot F'(x_n) + \mathcal{O}(dt^{3/2})$$

Combining these results, we get

$$\frac{dt}{4D}\left[\frac{(x_{n+1}-x_n)}{dt} - F(\bar{x})\right]^2 = \frac{dt}{4D}\left[\frac{x_{n+1}-x_n}{dt} - F(x_n)\right]^2 - dt\cdot\alpha\cdot F'(x_n) + \mathcal{O}(dt^{3/2})$$

See how this term cancells with such term in the Action.

$$\alpha\int dt\, F'(\bar{x})$$

Then we get

$$S = \underbrace{\frac{dt}{4D} \sum_{n=0}^{N-1} \left[\frac{x_{n+1}-x_n}{dt} - F(x_n) \right]^2}_{\text{the Action for } \alpha = 0 \text{ choice (Itô).}} + O(dt^{3/2})$$

This way all values of α give the same discretized Action, hence same statistics!

The subtlety comes when we write the Action as

$$S = \frac{1}{4D} \int dt \left(\dot{x} - F(x) \right)^2$$

and interpret integration/differentiation with standard calculus.

Only $\alpha = \frac{1}{2}$ (Stratonovich) is centrally/unbiased discretization thus matches with usual calculus.

Other α is off-centered discretization and does not match with usual calculus.

(We need to follow stochastic calculus).

[An easy example given on next page]

Bottom line: fix your convention (α) and stick to corresponding calculus.

For physicist, $\alpha = \frac{1}{2}$ (Stratonovich) is a preferred choice, where

$$S = \frac{1}{4D} \int_0^T dt \left(\dot{x} - F(x) \right)^2 + \frac{1}{2} \int_0^T dt \cdot F'(x)$$

and we stick to usual calculus. $\alpha = \frac{1}{2}$

$\alpha = \frac{1}{2}$ choice is invariant under time reversal.

Important: All these subtleties are because

$$dt \cdot \eta(t) = dW_t \simeq O(\sqrt{dt}).$$

Remark: ~~[illegible]~~ For multiplicative noise different α gives different statistics.

Further reading: ① Alain McKane, stochastic processes.
② Zinn-Justin, Quantum Field theory.

Remark: difference of stochastic calculus by an example.

$$\boxed{\int_0^T dW_t \cdot W_t = \frac{1}{2}\left(W_T^2 - W_0^2\right) \quad \text{usual calculus.}}$$

$$\int_0^T dW_t \cdot W_t = \sum_{n=0}^{N-1} \left(W_{n+1} - W_n\right)\left(\alpha W_{n+1} + (1-\alpha) W_n\right)$$

$$= \sum_{n=0}^{N-1} \left(W_{n+1} - W_n\right)\left\{\frac{1}{2}\left(W_{n+1} + W_n\right) + \frac{2\alpha - 1}{2}\left(W_{n+1} - W_n\right)\right\}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1}\left(W_{n+1}^2 - W_n^2\right) + \frac{2\alpha-1}{2}\sum_{n=0}^{N-1}\left(W_{n+1} - W_n\right)^2$$

$$= \frac{1}{2}\left(W_T^2 - W_0^2\right) + \frac{2\alpha - 1}{2} \cdot \sum_{n=0}^{N-1}\left[2D\,dt + \mathcal{O}\left(dt^{3/2}\right)\right]$$

(* for non-random W, this would be dt^2)

$$\Rightarrow \boxed{\int_0^T dW_t \cdot W_t \xrightarrow[\text{calculus}]{\text{stoch}} \frac{1}{2}\left(W_T^2 - W_0^2\right) + D\cdot(2\alpha - 1)\cdot T}$$

See how for Stratonovich ($\alpha = \frac{1}{2}$) both calculus give same result!

A canonical representation of the path integral: (we stick to Stratonovich.)

using an identity

$$\int \frac{dp}{2\pi i}\, e^{dt\cdot\left[Dp^2 - p\cdot x\right]} = \frac{1}{\sqrt{4\pi D\, dt}} \cdot e^{-\frac{dt}{4D}\cdot x^2}$$

we write the path integral in Eq. (2) as

$$G_T(x_T|x_0) = \lim_{\substack{dt\to 0\\ N\to\infty}} \cdot \int \prod_{k=1}^{N-1} dx_k \prod_{j=1}^{N} \frac{dp_j}{2\pi i}\, e^{-dt\sum_{n=0}^{N-1} p_n\left(\frac{x_{n+1}-x_n}{dt}\right)}$$

$$\times\, e^{dt\sum_{n=0}^{N-1}\left[D p_n^2 + p_n F\left(\frac{x_{n+1}+x_n}{2}\right)\right]}$$

$$\times\, e^{-dt\cdot\frac{1}{2}\cdot\sum_{n=0}^{N-1} F'\left(\frac{x_{n+1}+x_n}{2}\right)}$$

$$\equiv \int \mathcal{D}[x,p]\, e^{-S[x,p]}$$

with

$$S[x,p] = \int dt \left\{ p\,\dot{x} - \underbrace{\left[Dp^2 + pF - \tfrac{1}{2}F'\right]}_{\text{effective Hamiltonian.}} \right\}$$