How to get Stable distributions using RG ? ①

[functional RG]

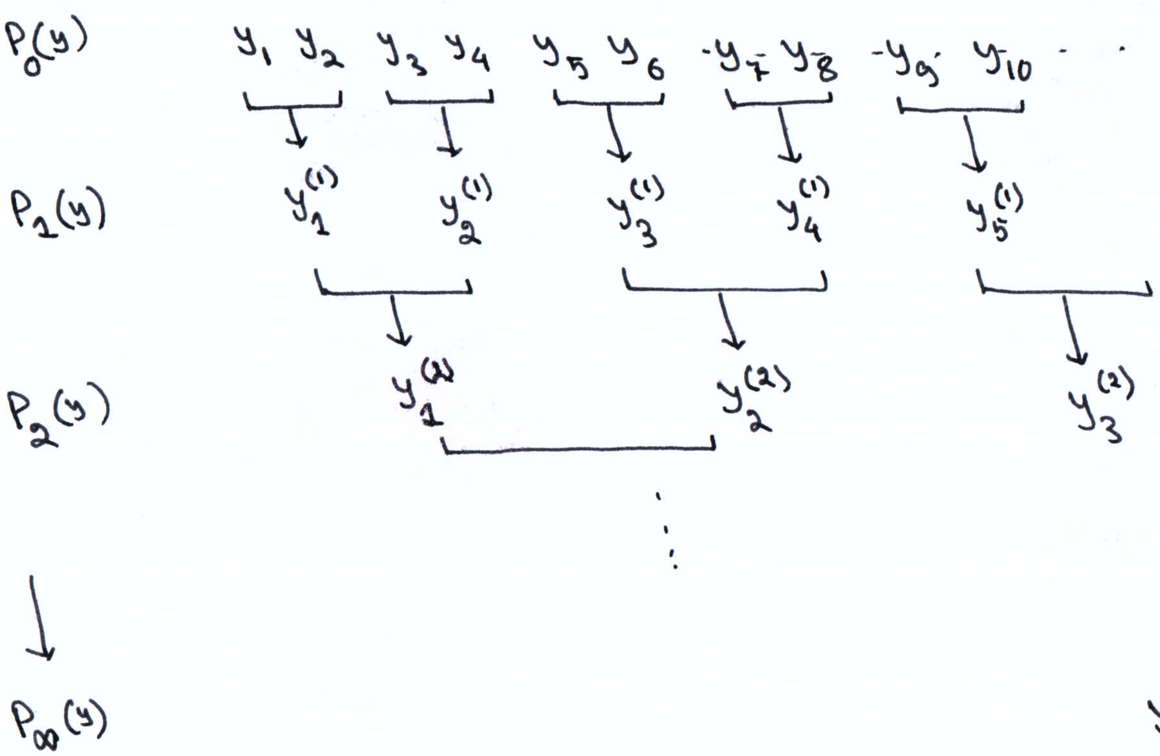
<u>Ref 1.</u> Exercise 12.11 in Book of Sethna.

Term papers:

<u>Ref 2.</u> "RG and probability theory", Jona-Lasinio, Phys. Rep. 352 (2001) 439.

<u>Ref 3.</u> "An elementary renormalization-group approach to ....". Ariel Amir, J Stat mech (2020) 013214.

<u>Basic idea:</u> For CLT case.



choose.

$$\boxed{\begin{aligned}\text{Mean } \langle y \rangle &= 0 \\ \langle y^2 \rangle &= 1\end{aligned}}$$

$$y = \frac{y_1 + y_2}{\sqrt{2}}$$

$$y^{(s)} = \frac{y_1 + \cdots + y_n}{\sqrt{n}}, \quad n = 2^s$$

Show that

$$P_\infty(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$$

(For CLT-case)

(take $\sigma^2 = 1$ for simplicity)

②

RG operation:

$$y = \frac{y_1 + y_2}{\sqrt{2}}$$ coarse-graining

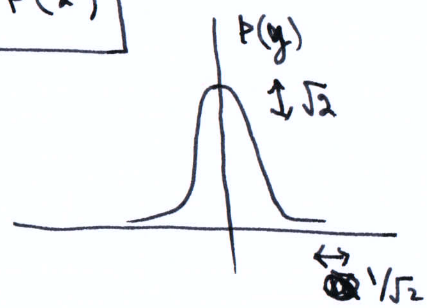
$$p_1(y) = \int dy_1\, dy_2\, p_0(y_1)\, p_0(y_2)\, \delta\left(\frac{y_1+y_2}{\sqrt{2}} - y\right)$$

$$= \sqrt{2} \int dy_1\, dy_2\, p_0(y_1)\, p_0(y_2)\, \delta(y_1 + y_2 - \sqrt{2}\, y)$$

$$= \sqrt{2} \int dy_2\, p_0(\sqrt{2}\cdot y - y_2)\, p_0(y_2)$$

$$\Rightarrow \boxed{R[p](y) = \sqrt{2} \int dx\, p(\sqrt{2}\cdot y - x)\, p(x)}$$

Rescaling



$$p_0 \xrightarrow{R} p_1 \xrightarrow{R} p_2 \cdots\cdots \xrightarrow{R} p_n$$

$$R^n[p](y)$$

A fixed point:

$$R[p_n^*](y) = p_{n+1}^*(y) = p_n^*$$

$$\boxed{p^*(y) = \sqrt{2} \int dx\, p^*(\sqrt{2}\, y - x)\, p^*(x)}$$ The RG equation.

Easier to solve in Fourier-space. $\hat{g}(k) = \int dy\, e^{iky}\, p(y)$

$$\Rightarrow \boxed{\hat{g}(k) = \left[\hat{g}\left(\frac{k}{\sqrt{2}}\right)\right]^2} \Rightarrow \hat{g}(k) = e^{-\frac{k^2}{2}}$$ is a fixed point

$\Updownarrow$

$p^*(x) =$ Gaussian.

* generalize :

$$y = \frac{y_1 + y_2}{b}$$

$$\to \quad p^*(y) = b \int dx \, p^*(x) \, p^*(by - x)$$

b is the rescaling factor.

General : $b = 2^{1/\alpha}$

$$\hat{g}(k) = \left[\hat{g}\left(\frac{k}{2^{1/\alpha}} \right) \right]^2$$

$P\left(\frac{M_n}{n^{1/\alpha}} = y\right) =$ Lévy distr. $\Rightarrow \quad \hat{g}(k) = e^{-|k|^\alpha}$ is a fixed point. $\Leftrightarrow$ Symmetric Lévy distr.

[choosing b is equivalent to ~~chos~~ keeping conserved quantities in RG]

[Information of b comes from your parent $p_0(x)$, knowing what is your typical fluctuations]

If be chosen incorrectly, the limiting distribution may be trivial $(\delta(x))$ or it may not exist !

Then, choice of b comes from demanding that their is a unique limit $p^*(y)$.

Is it an attractive fixed point ? [Relevant, marginal, irrelevant perturbations.]

linear stability analysis :

make a small peturbation around $p^*(y)$

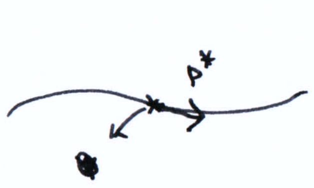
$$R[p^* + \epsilon \delta p](y) = b \int dx \, [p^* + \epsilon \delta p](by - x) \, (p^*(x) + \epsilon \delta p(x))$$

$$= p^*(y) + \epsilon [\mathcal{L} \cdot \delta p](y)$$

$$[\mathcal{L} \cdot \delta p](y) = b \int dx \left[p^*(by - x) \cdot \delta p(x) + \delta p(by - x) \cdot p^*(x) \right]$$

Eigenbasis :

$$[\mathcal{L} \cdot \psi_\lambda](y) = \lambda \, \psi_\lambda(y)$$



Take perturbation along any one eigen basis: $\delta p = \Psi_\lambda$

$$R[p^* + \epsilon \Psi_\lambda] = p^* + \epsilon \lambda \Psi_\lambda \Rightarrow \text{Repeated use: } \longrightarrow p^* + \epsilon \lambda^n \Psi_\lambda$$

$$\hat{R}[g^*(k) + \epsilon \hat{\Psi}_\lambda(k)] = g^*(k) + \epsilon \cdot \lambda \cdot \hat{\Psi}_\lambda(k)$$

⊛ What are eigenvalues and eigenvectors: fourier basis.

$$[\mathcal{L} \cdot \Psi_\lambda](y) = \lambda \Psi_\lambda(y)$$

$$[\hat{\mathcal{L}} \cdot \hat{\Psi}_\lambda](k) = \lambda \hat{\Psi}_\lambda(k)$$

$$\boxed{2 \cdot g^*\left(\frac{k}{b}\right) \hat{\Psi}_\lambda\left(\frac{k}{b}\right) = \lambda \hat{\Psi}_\lambda(k)}$$

$\boxed{\text{Solution}}$: $\hat{\Psi}_r(k) = (ik)^r g^*(k)$ with $\lambda = \frac{2}{b^r}$

What it means:

$\lambda_0 = 2 > 1 \longleftrightarrow$ relevant ← but does not conserve probability.

$\lambda_1 = \frac{2}{b} > 1 \longleftrightarrow$ "

for $b = \sqrt{2}$

$\lambda_2 = \frac{2}{b^2} = 1 \longleftrightarrow$ marginal

$\lambda_3 = \frac{2}{b^3} < 1 \longleftrightarrow$ irrelevant.

⋮

→ We don't perturb along this direction, because it breaks conservation that

$\langle y \rangle$, $\langle y^2 \rangle$ are fixed!