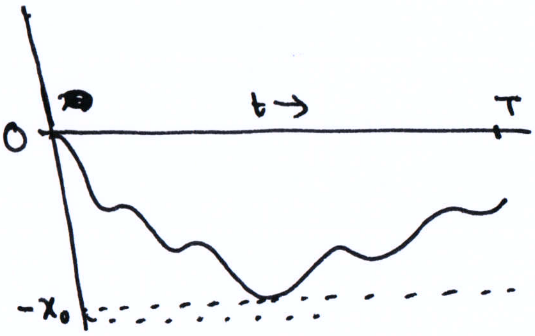
* Probability of maximum/minimum



Let $\mathcal{S}_T(-x_0)$ be the prob for a Brownian particle to have minimum position $(-x_0)$, with $x_0 > 0$, in time T. (By symmetry it is also the probability for maximum ~~val~~ position x_0).

Following a similar arguement as for the first passage prob, convince yourself that

$$\mathcal{S}_T(-x_0) = \frac{d S_T(x_0)}{d x_0} = \frac{e^{-\frac{x_0^2}{4Ddt}}}{\sqrt{\pi D dt}}$$

Remark: Method of images/reflection principle is a powerful, intuitive method. ~~Solution~~ Instead of an absorbing wall, if there is a reflecting wall at $x=0$, then argue ~~from~~ that the solution is

~~Rem~~

$$G_t(x|x_0) = \frac{1}{\sqrt{4\pi D t}}\left\{ e^{-\frac{(x-x_0)^2}{4Dt}} + e^{-\frac{(x+x_0)^2}{4Dt}} \right\}$$

$\circledast$

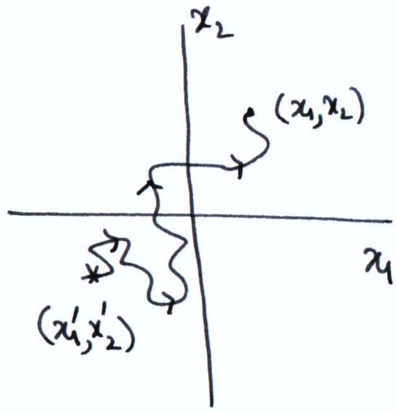
You can verify that this is a solution of $\frac{\partial G}{\partial t} = D\frac{\partial^2 G}{\partial x^2}$ with zero current condition at $x=0$ (reflecting wall)

$$j(x=0) = -D\frac{\partial G}{\partial x}\Big|_{x=0}$$

It is straight forward to generalize in presence of multiple reflecting/absorbing wall. [Think of method of images in electrostatics]

An application of reflection principle (and Bethe ansatz)

Brownian motion in 2D



$$G_t(x_1, x_2 | x_1', x_2') = g_t(x_1|x_1') \cdot g_t(x_2|x_2')$$

$$\downarrow$$

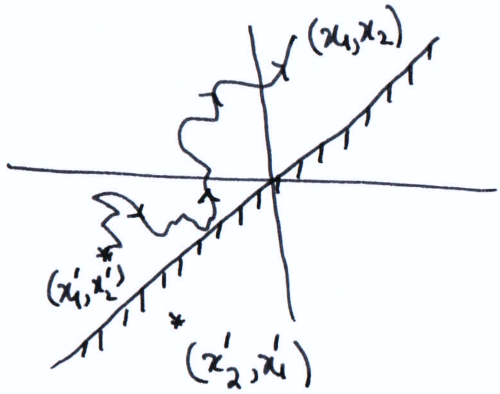
$$\frac{1}{\sqrt{4\pi D t}} \cdot e^{-\frac{(x_1 - x_1')^2}{4Dt}}$$

It is a solution of

$$\frac{\partial G}{\partial t} = \nabla^2 G \qquad \text{with vanishing } G \text{ at } (x_1, x_2) \to \infty.$$

Now consider a reflecting wall along $x_1 = x_2$ and start at $x_1' < x_2'$. This will confine the BM in the sector $x_1 < x_2$.

Use reflection principle to show that (for $x_1 < x_2$)



$$G_t(x_1, x_2 | x_1', x_2') = g_t(x_1|x_1') g_t(x_2|x_2')$$
$$+ g_t(x_1|x_2') g_t(x_2|x_1')$$

This satisfies diffusion equation with reflecting boundary ~~cdn~~ condition

$$J_\perp = -\hat{n} \cdot \vec{\nabla} G \Big|_{\text{wall}} = 0 \qquad \text{where } \hat{n} \text{ is perpendicular unit vector to the wall.}$$

Generalize this to d-dimension, where the BM is restricted to the sector (Weyl chamber) $x_1 < x_2 < \cdots < x_d$.

$$G_t(\bar{x} | \bar{x}') = \sum_{\sigma} g_t(x_1 | x'_{\sigma(1)}) \cdots g_t(x_d | x'_{\sigma(d)})$$

σ ← all permutations of $\{x_1', \ldots, x_d'\}$

Verify that this solves diffusion equation in d-dimension with reflecting boundary along the wall $x_i = x_{i+1}$ for $i = 1, 2, \cdots, d-1$.

~~Note how we write the~~

Remark: For an absorbing wall the solution is

$$G_t(\bar{x} \mid \bar{x}') = \sum_{\sigma} \underbrace{\epsilon_{\sigma(1), \cdots, \sigma(d)}}_{\substack{\text{Levi-civita} \\ \text{or} \\ \text{Sign of permutation.}}} \cdot g_t(x_1 \mid x'_{\sigma(1)}) \cdots g_t(x_d \mid x'_{\sigma(d)})$$

Remark: the two solutions describe probability of d-interacting particles in one-dimension with hard-core repulsion (reflection) or exclusion (absorption).

Note, the solution is expressed in terms of single particle solutions $g_t(x \mid x')$ and their product under permutations. This is the simplest example of Bethe ansatz.

Brownian functionals: [Ref: "Brownian functionals in physics and computer science", by Satya N Majumdar]

Often it is of interest to study observables which are functionals of Brownian path. For example,

(a) $t_+ = \int_0^T dt \cdot \Theta(x(t))$ time spent on positive half.

(b) $A = \int_0^T dt\, |x(t)|$ area under a Brownian curve. (of interest in economics)

also relevant in disordered quantum wires.

(c) $h = \int_0^T dt\, x(t) \cdot \Theta(x(t) - x_0)$ cumulative excess temperature beyond x_0, when $x(t)$ is daily temperature. In environmental science this is called "heating degree days".

(d) $p = \int_0^T dt \cdot e^{-\beta x(t)}$ integrated stock price upto some target time. (* typically stock prices are modeled by exponential of Brownian motion)

In general, we denote

$$f = \int_0^T dt\, V(x(t))$$ as Brownian functional.

We want to find probability distribution of f.

See how path-integral is a natural choice for studying such Brownian functionals.

Feynman - Kac formula : let $P_{x_0}(f,T)$ = the probability ~~for~~ of f for a Brownian ~~pat~~ motion started at x_0 and evolved upto time T.

Kac used Feynmans path integral idea to analye this probability and it is now known as the Feynman-Kac approach.

the general idea :

The generating function

$$R_{x_0}(\lambda,T) = \langle e^{-\lambda f} \rangle = \int df \, e^{-\lambda f} \, P_{x_0}(f,T)$$

Using path integral

$$R_{x_0}(\lambda,T) = \int dx_T \int_{x_0}^{x_T} \mathcal{D}[x] \, e^{-\int_0^T dt \left\{ \frac{\dot{x}^2}{4D} + \lambda V(x(t)) \right\}}$$

$$= \int dx_T \, G_T(x_T | x_0)$$

where we know

$$\boxed{\partial_t G_t(x|x_0) = D \, \partial_x^2 G_t - \lambda V(x) G_t + \delta(t)\,\delta(x-x_0)}$$

this is known as Feynman-Kac formula.

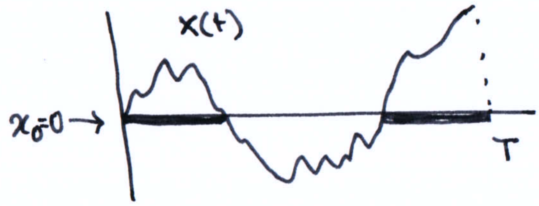
For finding $P_{x_0}(f,T)$, (1) solve for $G_t(x|x_0)$ with correct boundary condition

(2) then get $R_{x_0}(\lambda,T)$

(3) and finally perform the inverse Laplace transform of R to get $P_{x_0}(f,T)$.

An explicit example:

$$f = \int_0^T dt \cdot \Theta(x(t)) = \text{net time spent on the positive half.}$$



corresponding equation to be solved

$$\partial_t G = D\,\partial_x^2 G - \lambda\,\Theta(x) + \delta(t)\,\delta(x)$$

with boundary condition that $G_t(x) = 0$ for $x \to \pm\infty$.

(we denote $G_t(x|0) \equiv G_{t,\lambda}(x)$)

We shall remove the $\delta(t)\,\delta(x)$ term by explicitely demanding condition that $G_0(x) = \delta(x)$ and solving $G_t(x)$ for $t > 0$.

A standard tool: Laplace transformation.

Define
$$\tilde{G}_{s,\lambda}(x) = \int_0^\infty dt\, e^{-st}\, G_{t,\lambda}(x)$$

This gives, using

$$\int_0^\infty dt\, e^{-st}\left(-s + \partial_t\right) G_{t,\lambda}(x) = \int_0^\infty dt \cdot \frac{d}{dt}\left[e^{-st}\, G_{t,\lambda}(x)\right]$$

$$\Rightarrow -s\,\tilde{G}_{s,\lambda} + D\,\tilde{G}''_{s,\lambda} - \lambda\,\Theta(x)\,G_{s,\lambda} = -G_{0,\lambda}(x) = -\delta(x)$$

$$\Rightarrow \boxed{D\,\tilde{G}''_{s,\lambda} - \left(s + \lambda\,\Theta(x)\right)\tilde{G}_{s,\lambda} = -\delta(x)}$$

with $\tilde{G}_{s,\lambda}(x) = 0$ for $x \to \pm\infty$.

Verify that solution

$$\tilde{G}_{s,\lambda}(x) = \frac{\sqrt{s+\lambda}-\sqrt{s}}{\lambda\sqrt{D}}\; e^{-|x|\sqrt{\frac{s+\lambda\theta(x)}{D}}}$$

To get probability, use the definition

$$\int_0^T d\xi\, e^{-\lambda\xi}\, P_0(\xi,T) = \int dx_T\, G_{T,\lambda}(x_T)$$

$$\Rightarrow \boxed{\int_0^\infty dT\, e^{-sT}\int_0^T d\xi\, e^{-\lambda\xi}\, P_0(\xi,T)} = \int_{-\infty}^{\infty} dx_T \int_0^\infty dT\, e^{-sT}\, G_{T,\lambda}(x_T)$$

$$= \int_{-\infty}^{\infty} dx_T\, \tilde{G}_{s,\lambda}(x_T)$$

$$= \frac{1}{\sqrt{s(s+\lambda)}}$$

Inverting the double laplace transformation gives

$$\boxed{P_0(\xi,T) = \frac{1}{\pi\sqrt{\xi(T-\xi)}}}$$

How to do the inverse double laplace transformation?

see equation (59) − (64) of arxiv:2103.09032

uses Sokhotski-Plemlj formula of complex analysis.

Remarks: (1) Note that the probability does not depend on D.

This is because Brownian motion is invariant under a scale transformation

$$(x,t) \longrightarrow \left(\frac{x}{\sqrt{D}}, \frac{t}{D}\right).$$

You can see this from invariance of probability

$$P_T(x)\,dx = \frac{e^{-x^2/4DT}}{\sqrt{4\pi DT}}\,dx$$

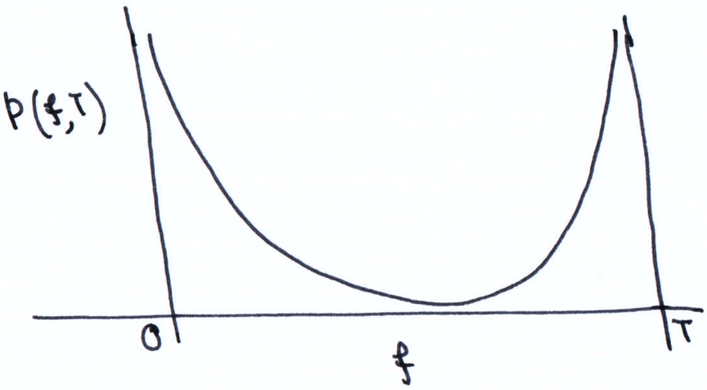
to be at $(x, x+dx)$.

(2) Cumulative probability of f is

$$F_T(x) = \int_0^x df \cdot P_0(f,T) = \frac{2}{\pi}\arcsin\left(\sqrt{\frac{x}{T}}\right)$$

This is ~~the~~ a famous arcsin-law of Brownian motion.

It is famous because the result is non intuitive



It is more probable that the Brownian motion spends most of its time ~~either~~ on ~~side~~ one side of the origin. Important in finance, games, stochastic thermodynamics etc.

Ref: see arxiv:2103.09032 and references there in.

(3) Prob of the time of last crossing of origin and prob of the time when the path achieved it's maximum also have EXACTLY sam distribution. These are known as Lévy's three arcsine-laws.

(4) For more examples of path integrals and Brownian Functionals see "Brownian Functionals in Physics and Computer science" by Satya N Majumdar.