Multi-time correlations

1. Correlations from fluctuating hydrodynamics

Define $\epsilon = \frac{1}{\sqrt{L}}$. The fluctuating hydrodynamics

$$\partial_t \rho(x,t) - \partial_x \left[D(\rho(x,t)) \partial_x \rho(x,t) \right] = \epsilon \, \partial_x \left[\sqrt{\sigma(\rho(x,t))} \, \eta(x,t) \right] \tag{1}$$

with covariance

$$\langle \eta(x,t) \eta(y,t') \rangle = \delta(x-y) \delta(t-t') \tag{2}$$

1.1. Series expansion:

$$\rho(x,t) = \overline{\rho}(x) + \epsilon \, r_1(x,t) + \frac{\epsilon^2}{2!} r_2(x,t) + \cdots \tag{3}$$

Leads to

$$D(\rho) = D(\overline{\rho}) + \epsilon \, D'(\overline{\rho}) r_1 + \frac{\epsilon^2}{2!} \left[r_1^2 D''(\overline{\rho}) + r_2 D'(\overline{\rho}) \right] + \cdots \tag{4}$$

$$\sqrt{\sigma(\rho)} = \sqrt{\sigma(\overline{\rho})} + \frac{\epsilon}{2} \frac{\sigma'(\overline{\rho})}{\sqrt{\sigma(\overline{\rho})}} \, r_1 + \cdots \tag{5}$$

and

$$D(\rho) \partial_x \rho = D(\overline{\rho}) \partial_x \overline{\rho} + \epsilon \, \partial_x \left[D(\overline{\rho}) r_1 \right] + \frac{\epsilon^2}{2} \partial_x \left[D(\overline{\rho}) r_2 + D'(\rho) r_1^2 \right] + \cdots \tag{6}$$

Substituting in the fluctuating hydrodynamics equation (1) one gets

(i) Zeroth order

$$\partial_t \overline{\rho} - \partial_x^2 \left[D(\overline{\rho}) \partial_x \overline{\rho} \right] = 0 \tag{7}$$

(ii) Linear order

$$\partial_t r_1 - \partial_x^2 \left[D(\overline{\rho}) r_1 \right] = \partial_x \left[\sqrt{\sigma(\overline{\rho})} \; \eta \right] \tag{8}$$

(iii) The ϵ^2 order

$$\partial_t r_2 - \partial_x^2 \left[D(\overline{\rho}) r_2 \right] = \partial_x^2 \left[D'(\overline{\rho}) \, r_1^2 \right] + \partial_x \left[\frac{\sigma'(\overline{\rho})}{\sqrt{\sigma(\overline{\rho})}} \, r_1 \, \eta \right] \tag{9}$$

Green's function and solution

$$\partial_t G(x,y,t) - \partial_x^2\left[D(\overline{\rho})G(x,y,t)\right] = 0 \qquad \text{with} \quad G(x,y,0)=\delta(x-y) \tag{10}$$

The solution

$$r_1(x,t) = -\int_{-\infty}^{t} ds \int_0^1 dz \partial_z G(x,z,t-s)\sqrt{\sigma(\overline{\rho}(z))}\,\eta(z,s) \tag{11}$$

$$r_2(x,t) = -\int_{-\infty}^{t} ds \int_0^1 dz \partial_z G(x,z,t-s)\left\{\frac{\sigma'(\overline{\rho}(z))}{\sqrt{\sigma(\overline{\rho}(z))}} r_1(z,s)\eta(z,s) + \partial_z\left[D'(\overline{\rho}(z))r_1^2(z,s)\right]\right\} \tag{12}$$

In writing we made an integration by parts in z variable.

Remark: By definition $\langle r_2(x,t)\rangle = 0$. In order to get this from the solution (12) one chooses

$$\langle r_1(x,t)\eta(x,t)\rangle = 0 \qquad \text{and} \qquad \partial_x\left[D'(\overline{\rho}(x))\left\langle r_1^2(x,t)\right\rangle\right] = 0 \tag{13}$$

This is consistent with another result $\langle J\rangle = -\langle D(\rho)\partial_x\rho\rangle = -D(\overline{\rho})\partial_x\overline{\rho}$ in (6). (** Is this an Ito-Stratonovich type choise?**)

1.2. Two-time correlation

$$\langle\rho(x_1,t_1)\rho(x_2,t_2)\rangle_c = \epsilon^2\,\langle r_1(x_1,t_1)r_1(x_2,t_2)\rangle + \mathrm{O}(\epsilon^3) = \epsilon^2\,c(x_1,t_1;x_2,t_2) + \mathrm{O}(\epsilon^3) \tag{14}$$

where from (11) one gets

$$c(x_1,t_1;x_2,t_2) = \int_{-\infty}^{\min(t_1,t_2)} ds \int_0^1 dz \sigma(\overline{\rho}(z))\left[\partial_z G(x_1,z,t_1-s)\right]\left[\partial_z G(x_2,z,t_2-s)\right] \tag{15}$$

1.3. Three-time correlation

$$\begin{aligned}\langle\rho(x_1,t_1)\rho(x_2,t_2)\rho(x_3,t_3)\rangle_c &= \epsilon^3\,\langle r_1(x_1,t_1)r_1(x_2,t_2)r_1(x_3,t_3)\rangle \\ &+\frac{\epsilon^4}{2}\Big[\langle r_1(x_1,t_1)r_1(x_2,t_2)r_2(x_3,t_3)\rangle + (2\leftrightarrow 3) + (1\leftrightarrow 3)\Big] + \cdots\end{aligned} \tag{16}$$

where $(i\leftrightarrow j)$ denotes interchange of coordinate indices $(x_i,t_i)\leftrightarrow(x_j,t_j)$ in the first term inside square brackets. (The prefactor $\frac{1}{2}$ comes from the same prefactor of r_2 in (3).)

It is easy to check that the ϵ^3 term vanishes due to $\langle\eta\eta\eta\rangle = 0$.

For the ϵ^4 order term we use (11)-(12) and explicitly write

$$\begin{aligned}\langle\rho(x_1,t_1)\rho(x_2,t_2)\rho(x_3,t_3)\rangle_c \simeq -\frac{\epsilon^4}{2}&\int_{-\infty}^{t_1} ds_1 \int_0^1 dz_1 \int_{-\infty}^{t_2} ds_2 \int_0^1 dz_2 \int_{-\infty}^{t_3} ds_3 \int_0^1 dz_3 \\ &\partial_{z_1}G(x_1,z_1,t_1-s_1)\partial_{z_2}G(x_2,z_2,t_2-s_2)\partial_{z_3}G(x_3,z_3,t_3-s_3) \\ &\left[\omega(z_1,s_1;z_2,s_2;z_3,s_3) + \widehat{\omega}(z_1,s_1;z_2,s_2;z_3,s_3)\right]\end{aligned} \tag{17}$$

where

$$\omega = \sqrt{\sigma(\overline{\rho}(z_1))\sigma(\overline{\rho}(z_2))}\frac{\sigma'(\overline{\rho}(z_3))}{\sqrt{\sigma(\overline{\rho}(z_3))}}\,\langle\eta(z_1,s_1)\eta(z_2,s_2)\eta(z_3,s_3)r_1(z_3,s_3)\rangle$$

$$+(2 \leftrightarrow 3) + (1 \leftrightarrow 3)$$

$$\widehat{\omega} = \sqrt{\sigma(\overline{\rho}(z_1))\sigma(\overline{\rho}(z_2))}\; \partial_{z_3}\left[D'(\overline{\rho}(z_3))\left\langle \eta(z_1,s_1)\eta(z_2,s_2)r_1^2(z_3,s_3)\right\rangle\right] \\ +(2 \leftrightarrow 3) + (1 \leftrightarrow 3)$$

The average in ω and $\widehat{\omega}$ can be calculated using Wick's theorem

$$\langle \eta_1\eta_2\eta_3 r_1\rangle = \langle \eta_1\eta_3\rangle\langle \eta_2 r_1\rangle + \langle \eta_2\eta_3\rangle\langle \eta_1 r_1\rangle + \langle \eta_1\eta_2\rangle\langle \eta_3 r_1\rangle$$

Using (13) one finds $\langle \eta(z_3,s_3)r_1(z_3,s_3)\rangle = 0$ which makes the last term above vanish. From the rest of the terms and using (2) one gets

$$\omega = \sqrt{\sigma(\overline{\rho}(z_2))}\;\sigma'(\overline{\rho}(z_3))\,\langle \eta(z_2,s_2)r_1(z_3,s_3)\rangle\,\delta(z_1-z_3)\delta(s_1-s_3) \\ +\sqrt{\sigma(\overline{\rho}(z_1))}\;\sigma'(\overline{\rho}(z_3))\,\langle \eta(z_1,s_1)r_1(z_3,s_3)\rangle\,\delta(z_2-z_3)\delta(s_2-s_3) \\ +(2 \leftrightarrow 3) + (1 \leftrightarrow 3)$$

Writing in closed form

$$\omega = \sum_{perm(1,2,3)} \sqrt{\sigma(\overline{\rho}(z_2))}\;\sigma'(\overline{\rho}(z_3))\,\langle \eta(z_2,s_2)r_1(z_3,s_3)\rangle\,\delta(z_1-z_3)\delta(s_1-s_3) \tag{18}$$

where the sum is over all 6 permutations over the coordinate indices $(1,2,3)$.

For a similar calculation of $\widehat{\omega}$ one uses

$$\left\langle \eta_1\eta_2 r_1^2\right\rangle = 2\langle \eta_1 r_1\rangle\langle \eta_2 r_1\rangle + \langle \eta_1\eta_2\rangle\left\langle r_1^2\right\rangle$$

Then, using (13) to get $\partial_{z_3}\left[D'(\overline{\rho}(z_3))\left\langle r_1^2(z_3,s_3)\right\rangle\right] = 0$ which makes contribution from last term above vanish. This leads to

$$\widehat{\omega} = \sum_{perm(1,2,3)} \sqrt{\sigma(\overline{\rho}(z_1))\sigma(\overline{\rho}(z_2))}\; \partial_{z_3}\left[D'(\overline{\rho}(z_3))\,\langle \eta(z_1,s_1)r_1(z_3,s_3)\rangle\,\langle \eta(z_2,s_2)r_1(z_3,s_3)\rangle\right] \tag{19}$$

Using (11) one can easily check that the contribution in (17) from the term with ω in (18) is

$$\frac{\epsilon^4}{2}\sum_{perm(1,2,3)}\int_{-\infty}^{\min(t_1,t_3)} ds_3 \int_0^1 dz_3 \partial_{z_3}G(x_1,z_3,t_1-s_3)\partial_{z_3}G(x_3,z_3,t_3-s_3)c(x_2,t_2;z_3,s_3)\sigma'(\overline{\rho}(z_3)) \\ = I \tag{20}$$

where $c(x_2,t_2;x_3,t_3) = \langle r_1(x_2,t_2)r_1(x_3,t_3)\rangle$. Similarly contribution from the term with $\widehat{\omega}$ in (19) is

$$-\frac{\epsilon^4}{2}\sum_{perm(1,2,3)}\int_{-\infty}^{t_3} ds_3 \int_0^1 dz_3 \partial_{z_3}G(x_3,z_3,t_3-s_3)\partial_{z_3}\left[D'(\overline{\rho}(z_3))c(x_1,t_1;z_3,s_3)c(x_2,t_2;z_3,s_3)\right] \\ = \widehat{I} \tag{21}$$

Then the three-time correlation is sum of the above two terms.

$$\langle \rho(x_1,t_1)\rho(x_2,t_2)\rho(x_3,t_3)\rangle_c \simeq I + \widehat{I} \tag{22}$$

1.4. Equal-time correlation

Using (15) one gets the equal-time correlation

$$c(x_1,x_2)=c(x_1,0;x_2,0)=\int_0^\infty ds\int_0^1 dz_1\sigma(\overline{\rho}(z_1))\left[\partial_{z_1}G(x_1,z_1,s)\right]\left[\partial_{z_1}G(x_2,z_1,s)\right]$$

where we changed the integration variable s. Using integration by parts and introducing another integration variable z_2 by using

$$\frac{d}{dz_1}G(x_2,z_1,s)=\int_0^1 dz_2 G(x_2,z_2,s)\frac{d}{dz_1}\delta(z_2-z_1)$$

one can rewrite

$$c(x_1,x_2)=\int_0^\infty ds\int_0^1 dz_1\int_0^1 dz_2 G(x_1,z_1,s)G(x_2,z_2,s)\;\Omega(z_1,z_2) \tag{23}$$

where

$$\Omega(z_1,z_2)=\frac{d}{dz_1}\left[\sigma(\overline{\rho}(z_1))\delta'(z_2-z_1)\right]=\frac{1}{2}\sum_{perm(1,2)}\frac{d}{dz_1}\left[\delta'(z_2-z_1)\sigma(\overline{\rho}(z_1))\right] \tag{24}$$

(compare with (27c).)

An advantage of this form is that derivation of the differential equation for the correlation is simple. Using (10) one gets

$$\begin{aligned}\partial_{x_1}^2&\left[D(\overline{\rho}(x_1))c(x_1,x_2)\right]+\partial_{x_2}^2\left[D(\overline{\rho}(x_2))c(x_1,x_2)\right]\\ &=\int_0^1 dz_1\int_0^1 dz_2\int_0^\infty ds\frac{d}{ds}\left[G(x_1,z_1,s)G(x_2,z_2,s)\right]\;\Omega(z_1,z_2)\\ &=-\Omega(x_1,x_2)\end{aligned} \tag{25}$$

Three point correlation: We show below that the three point correlation can also be written in a form similar to (23), more precisely,

$$\langle\rho(x_1,0)\rho(x_2,0)\rho(x_3,0)\rangle_c\simeq\epsilon^4\;c(x_1,x_2,x_3) \tag{26}$$

where

$$c(x_1,x_2,x_3)=\int_0^\infty ds\int_0^1 dz_1\int_0^1 dz_2\int_0^1 dz_3 G(x_1,z_1,s)G(x_2,z_2,s)G(x_3,z_3,s)\Omega(z_1,z_2,z_3) \tag{27a}$$

(Note, there is only one integration over time.) This way it is simple to check (like in (25)) that

$$\begin{aligned}\partial_{x_1}^2&\left[D(\overline{\rho}(x_1))c(x_1,x_2,x_3)\right]+\partial_{x_2}^2\left[D(\overline{\rho}(x_2))c(x_1,x_2,x_3)\right]+\partial_{x_3}^2\left[D(\overline{\rho}(x_3))c(x_1,x_2,x_3)\right]\\ &=-\Omega(x_1,x_2,x_3)\end{aligned} \tag{27b}$$

We show that

$$\Omega(z_1,z_2,z_3)=\frac{1}{2}\sum_{perm(1,2,3)}\Big\{\partial_{z_3}^2\left[D'(\overline{\rho}(z_3))c(z_1,z_3)c(z_2,z_3)\right]+\partial_{z_3}\left[\delta'(z_1-z_3)c(z_2,z_3)\sigma'(\overline{\rho}(z_3))\right]\Big\} \tag{27c}$$

Derivation: Setting $t_1 = t_2 = t_3 = 0$ in (20) one gets

$$I = \frac{\epsilon^4}{2} \sum_{perm(1,2,3)} \int_0^\infty ds \int_0^1 dz_3 \left[\partial_{z_3} G(x_1, z_3, s)\right] \left[\partial_{z_3} G(x_3, z_3, s)\right] c(x_2, 0; z_3, -s) \sigma'(\overline{\rho}(z_3)) \quad (28)$$

where we made a change of variable $s_3 \to -s$. Further simplification comes from the following two equations

$$c(x_2, 0; z_3, -s) = \int_0^1 dz_2 G(x_2, z_2, s) c(z_2, z_3) \quad (29)$$

$$\frac{d}{dz_3} G(x_1, z_3, s) = \int_0^1 dz_1 G(x_1, z_1, s) \frac{d}{dz_3} \delta(z_1 - z_3) \quad (30)$$

Substituting in (28) and using integration by parts over z_3 variable leads to

$$I = \frac{\epsilon^4}{2} \sum_{perm(1,2,3)} \int_0^\infty ds \int_0^1 dz_1 dz_2 dz_3 G(x_1, z_1, s) G(x_2, z_2, s) G(x_3, z_3, s) \frac{d}{dz_3} \left[\sigma'(\overline{\rho}(z_3)) c(z_2, z_3) \delta'(z_1 - z_3)\right] \quad (31)$$

Similarly, setting $t_1 = t_2 = t_3 = 0$ in (21) and using integration by parts one gets

$$\widehat{I} = \frac{\epsilon^4}{2} \sum_{perm(1,2,3)} \int_0^\infty ds \int_0^1 dz_3 G(x_3, z_3, s) \frac{d^2}{dz_3^2} \left[D'(\overline{\rho}(z_3)) c(x_1, 0; z_3, -s) c(x_2, 0; z_3, -s)\right] \quad (32)$$

Further, substituting (29) one gets

$$\widehat{I} = \frac{\epsilon^4}{2} \sum_{perm(1,2,3)} \int_0^\infty ds \int_0^1 dz_1 dz_2 dz_3 G(x_1, z_1, s) G(x_2, z_2, s) G(x_3, z_3, s) \frac{d^2}{dz_3^2} \left[D'(\overline{\rho}(z_3)) c(z_1, z_3) c(z_2, z_3)\right] \quad (33)$$

Adding the two contributions $I + \widehat{I}$ one gets $(27a)$-$(27c)$.

Conjecture: One expects similar structure for higher order correlation

$$\langle\langle \rho(x_1) \cdots \rho(x_k) \rangle\rangle = \epsilon^{2k-2} \int_0^\infty ds \int_0^1 dz_1 \cdots \int_0^1 dz_k G(x_1, z_2, s) \cdots G(x_k, z_k, s) \Omega(z_1, \cdots, z_k) \quad (34)$$

It is easy to check (like in (25))

$$\sum_{n=1}^k \partial_{x_n}^2 D(\overline{\rho}(x_n)) \langle\langle \rho(x_1) \cdots \rho(x_k) \rangle\rangle = -\epsilon^{2k-2} \Omega(x_1, \cdots, x_k) \quad (35)$$

as obtained in the paper by Bertini et al JSP 2009 using macroscopic fluctuation theory; a derivation is given in the next section.

Remark: (need to check more carefully.) One can write multi-time correlation in a closed form. For $D(\rho) = 1$ one gets a general form

$$c(x_1, t_1; x_2, t_2) = \sum_{perm(1,2)} \int_0^{\min(t_1, t_2)} ds \int dz_1 \int dz_2 G(x_1, z_1, t_1 - s) G(x_2, z_2, t_2 - s) \Lambda(z_1, z_2)$$

with

$$\Lambda(z_1, z_2) = \frac{1}{2}\frac{d}{dz_1}\left[\delta'(z_1 - z_2)\sigma(z_1)\right]$$

The permutation is over only the (x, t) coordinates and not on (z, t).

For three point correlation

$$c(x_1, t_1; x_2, t_2; x_3, t_3) = \sum_{perm(1,2,3)} \int_0^{\min(t_1,t_2)} ds \int dz_1 \int dz_2 \int dz_3 G(x_1, z_1, t_1 - s) \\ G(x_2, z_2, t_2 - s)G(x_3, z_3, t_3 - s)\Lambda(z_1, z_2, z_3)$$

with

$$\Lambda(z_1, z_2, z_3) = \frac{1}{2}\frac{d}{dz_3}\left[\delta'(z_3 - z_1)\sigma'(z_3)c(z_2, z_3)\right]$$

2. Derivation using Hamilton-Jacobi method

The large deviation function $\phi[\rho]$ of a density fluctuation is defined as

$$P[\rho(x)] \sim e^{-L\phi[\rho(x)]}$$

Using macroscopic fluctuation theory one shows that the large deviation function follows

$$\int_0^1 dx \left\{\frac{1}{2}\sigma(\rho)\left(\partial_x \frac{\delta\phi}{\delta\rho}\right)^2 - (\partial_x\rho)\, D(\rho)\left(\partial_x\frac{\delta\phi}{\delta\rho}\right)\right\} = 0 \tag{36}$$

This is the Hamilton-Jacobi equation for minimal action (Bertini et al 2001).

The generating function of density correlation is defined by

$$\mu[h] = \int_0^1 dx h(x)\overline{\rho}(x) + \frac{1}{2!}\int_0^1 dx_1 dx_2 c(x_1, x_2) + \frac{1}{3!}\int_0^1 dx_1 dx_2 dx_3 c(x_1, x_2, x_3) + \cdots \tag{37}$$

The $\mu[h]$ and $\phi[\rho]$ are related by Legendre transformation.

$$\mu[h] = -\phi[\rho] + \int_0^1 dx h(x)\rho(x)$$

where

$$h(x) = \frac{\delta\phi[\rho]}{\delta\rho(x)} \qquad \text{and} \qquad \rho(x) = \frac{\delta\mu[h]}{\delta h(x)} \tag{38}$$

Substituting (38) in (36) one gets

$$\int_0^1 dx \left\{\frac{1}{2}\sigma\left(\frac{\delta\mu}{\delta h(x)}\right)(\partial_x h)^2 - \left(\partial_x\frac{\delta\mu}{\delta h}\right) D\left(\frac{\delta\mu}{\delta h(x)}\right)(\partial_x h)\right\} = 0 \tag{39}$$

Correlations: The differential equations followed by correlations can be obtained by substituting the expansion (37) in (39) and expand in orders of $h(x)$.

In the leading order one gets

$$\int_0^1 dx h(x)\partial_x\left[D(\overline{\rho}(x))\partial_x\overline{\rho}(x)\right]=0 \tag{40}$$

where we used integration by parts. The solution $D(\overline{\rho}(x))\partial_x\overline{\rho}(x)=J$ determines $\overline{\rho}(x)$.

In the second order

$$\int_0^1 dx_1dx_2h(x_1)h(x_2)\left\{\frac{1}{2}\partial_{x_1}\partial_{x_2}\left[\sigma(\overline{\rho}(x_1))\delta(x_1-x_2)\right]+\partial_{x_1}^2\left[D(\overline{\rho}(x_1))c(x_1,x_2)\right]\right\}=0 \tag{41}$$

A possible reason (to get (25)) the integral vanishes is because the integrand is anti-symmetric under exchange of x_1 and x_2.

$$\int_0^1 dx_1dx_2h(x_1)h(x_2)f(x_1,x_2)=0 \qquad \text{if} \qquad f(x_1,x_2)=-f(x_2,x_1)$$

This leads to

$$\begin{aligned}\partial_{x_1}^2\left[D(\overline{\rho}(x_1))c(x_1,x_2)\right]+\partial_{x_2}^2\left[D(\overline{\rho}(x_2))c(x_2,x_1)\right]=&-\frac{1}{2}\partial_{x_1}\partial_{x_2}\left[\sigma(\overline{\rho}(x_1))\delta(x_1-x_2)\right]\\&-\frac{1}{2}\partial_{x_2}\partial_{x_1}\left[\sigma(\overline{\rho}(x_2))\delta(x_2-x_1)\right]\end{aligned}$$

Rewriting the right hand side one gets

$$\partial_{x_1}^2\left[D(\overline{\rho}(x_1))c(x_1,x_2)\right]+\partial_{x_2}^2\left[D(\overline{\rho}(x_2))c(x_1,x_2)\right]=\partial_{x_1}\left[\sigma(\overline{\rho}(x_1))\delta'(x_1-x_2)\right] \tag{42}$$

which is same as (25).

In the cubic order one follows similar argument. From (39) one gets

$$\int_0^1 dx_1dx_2dx_3h(x_1)h(x_2)h(x_3)f(x_1,x_2,x_3)=0 \tag{43}$$

where

$$\begin{aligned}f(x_1,x_2,x_3)=\partial_{x_1}^2\left[D(\overline{\rho}(x_1))c(x_1,x_2,x_3)\right]+\partial_{x_1}^2\left[D'(\overline{\rho}(x_1))c(x_1,x_2)c(x_1,x_3)\right]\\+\partial_{x_1}\left[\sigma'(\overline{\rho}(x_1))\delta'(x_1-x_2)c(x_1,x_2)\right]\end{aligned} \tag{44}$$

Now, one possible reason why the integral vanishes, is

$$\sum_{perm(1,2,3)} f(x_1,x_2,x_3)=0 \tag{45}$$

(for example, if $f(x_1,x_2,x_3)$ is odd under interchange of pairs.) This leads to

$$\begin{aligned}2\left\{\partial_{x_1}^2\left[D(\overline{\rho}(x_1))c(x_1,x_2,x_3)\right]+\partial_{x_2}^2\left[D(\overline{\rho}(x_2))c(x_1,x_2,x_3)\right]+\partial_{x_3}^2\left[D(\overline{\rho}(x_3))c(x_1,x_2,x_3)\right]\right\}\\=-\sum_{perm(1,2,3)}\left\{\partial_{x_1}^2\left[D'(\overline{\rho}(x_1))c(x_1,x_2)c(x_1,x_3)\right]+\partial_{x_1}\left[\sigma'(\overline{\rho}(x_1))\delta'(x_1-x_2)c(x_1,x_2)\right]\right\}\end{aligned}$$

This leads to the same equation as in (35).

3. Writing large deviation function for SEP in terms of Greens function

Given (34) can one write the known result of generating function $\mu[h]$ for symmetric exclusion process in terms of Greens functions?

Generating function

$$\mu[h] = \int_0^1 dx \left\{ \log\left(1 + F(e^h - 1)\right) - \log \frac{F'}{\rho_b - \rho_a} \right\} \tag{46}$$

with

$$F'' = \frac{(e^h - 1)(F')^2}{1 + F(e^h - 1)}; \qquad F(0) = \rho_a; \qquad F(1) = \rho_b \tag{47}$$

Equilibrium: for $\rho_a = \rho_b$ one gets $F = \rho$ and leads to

$$\mu_{eq}[h] = \int_0^1 dx \left\{ \log\left(1 + \rho(e^{h(x)} - 1)\right) \right\} \tag{48}$$

Equilibrium correlations

$$c(x, y) = \frac{\sigma(\rho)}{2}\delta(x - y); \qquad c(x, y, z) = \frac{\sigma(\rho)}{2}\frac{\sigma'(\rho)}{2}\delta(x - y)\delta(x - z) \tag{49}$$

$$c(x, y, z, w) = \frac{\sigma(\rho)}{2}\left[1 - 3\sigma(\rho)\right]\delta(x - y)\delta(x - z)\delta(x - w) \tag{50}$$

Question: can one reproduce this using Greens function? Is there a general formula of the equilibrium correlation for arbitrary D and σ?

Remark: In out of equilibrium one may and write

$$\mu[h] = \mu_0[h] + \int_0^1 dx \left\{ \log\left[\frac{1 + F(e^h - 1)}{1 + \overline{\rho}(x)(e^h - 1)}\right] - \log \frac{F'}{\rho_b - \rho_a} \right\} \tag{51}$$

where $\mu_0[h]$ contains the equilibrium part of the correlation which one can easily write as delta correlated. May be its best to write $F = \overline{\rho}(x) + \psi(x)$

Out of equilibrium Write $h \to \epsilon h$. An expansion

$$F(x) = \overline{\rho}(x) + \epsilon F_1(x) + \epsilon^2 F_2(x) + \cdots \tag{52}$$

One shows

$$F_1(x) = -(\rho_a - \rho_b)^2 \int dy h(y) U(x, y) \tag{53}$$

where

$$U(x, y) = \int_0^\infty ds G(x, y, s) \tag{54}$$

One may verify this from the equation for F_1 which is

$$F_1''(x) = h(x)\left[F_0'(x)\right]^2 \tag{55}$$

References