Multi-time correlations of current using the macroscopic fluctuation theory

Tridib Sadhu[1] **and Bernard Derrida**[2]

[1] Philippe Meyer Institute for Theoretical Physics, Physics Department, Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05 - France.

E-mail: `tridib.sadhu@lpt.ens.fr`

[2] Collège de France, 11 place Marcelin Berthelot, 75231 Paris Cedex 05 - France.

E-mail: `derrida@lps.ens.fr`

Abstract. Multi-time correlation of current for one dimensional diffusive system coupled with reservoirs has been recently calculated in j stat mech 2016. We present two alternate derivation of the correlations which are logically independent of the earlier work. In one, we show how to obtain the correlation using the macroscopic fluctuation theory which provides a systematic way of calculating correlations of arbitrary order. The results are for general system in terms of the diffusivity and the mobility. In the second approach we verify the results in an exact microscopic solution of the symmetric exclusion process.

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1. Introduction

Understanding the steady state properties of non-equilibrium systems is an important topic in modern statistical physics. For interacting particle systems a general approach is based on a fluctuating hydrodynamics description [1, 2]. Such a description is the starting point of a recently developed macroscopic fluctuation theory [3–7] which has proved useful to calculate large deviation functions of diffusive systems. While a large amount of results have been obtained based on the fluctuating hydrodynamics approach, they are mostly about one-time statistics [1,2,8–21]. In this work, we use this framework to see how some other dynamical properties in a non-equilibrium steady state, namely multi-time correlations, spectral distribution, and linear-response can be calculated.

We consider the non-equilibrium steady state of a one-dimensional system coupled with reservoirs at two ends. It may be a system of interacting particles coupled with reservoirs at different chemical potentials, or a thermal conducting rod coupled with heat baths at different temperatures. One uses fluctuating hydrodynamics to describe such a system in the large system size L limit; in systems where diffusion is the transport mechanism, one defines hydrodynamics coordinates $x = \frac{X}{L}$ and $\tau = \frac{t}{L^2}$ where X and t are position and time, respectively. The time evolution of the system is described in terms of a hydrodynamic density $\rho(x,\tau)$ and a hydrodynamic current $J(x,\tau)$ which satisfy [1, 2]

$$\partial_\tau \rho(x,\tau) = -\partial_x J(x,\tau) \quad \text{with} \quad J(x,\tau) + D(\rho(x,\tau))\partial_x \rho(x,\tau) = \eta(x,\tau) \tag{1}$$

where $D(\rho)$ is the diffusivity. What (1) tells us is that on average the current follows Fick's law ($J = -D(\rho)\partial_x \rho$) and the fluctuations are modeled by a Gaussian random noise $\eta(x,\tau)$ with a covariance

$$\langle \eta(x,\tau)\eta(y,\tau')\rangle = \frac{1}{L}\sigma(\rho(x,\tau))\,\delta(x-y)\,\delta(\tau-\tau') \tag{2}$$

where $\sigma(\rho)$ is the mobility.

In this paper, we use (1)-(2) to derive the two-time correlations $\langle \rho(x,\tau)\rho(y,\tau')\rangle_c$ and $\langle J(x,\tau)J(y,\tau')\rangle_c$, and to obtain the correlations of the integrated current $q(x,\tau)$ which is defined as the time integral of $J(x,\tau)$. We will show in particular that

$$\langle q(x,\tau)q(x,\tau')\rangle_c \simeq \begin{cases} \dfrac{\min(\tau,\tau')}{L}\displaystyle\int_0^1 dz\sigma(\overline{\rho}(z)) & \text{for } \tau,\ \tau' \gg 1 \\ \dfrac{1}{L}\left[\sqrt{\tau}+\sqrt{\tau'}-\sqrt{|\tau-\tau'|}\right]\dfrac{\sigma(\overline{\rho}(x))}{2\sqrt{\pi\, D(\overline{\rho}(x))}} & \text{for } \tau,\ \tau' \ll 1 \end{cases} \tag{3}$$

where $\overline{\rho}(x)$ is the average density in the steady state. At large times the asymptotic behavior corresponds to a standard Brownian motion with known [22] variance whereas at short time the covariance is that of a fractional Brownian motion [23]. One can notice that the short time behavior also gives the correlation of the integrated current of an infinite system at equilibrium. (One may infer the time dependence from the correlation of the height fluctuations in the Edwards-Wilkinson interface growth [24].)

Our approach is valid for a general diffusive system in a non-equilibrium steady state for which the microscopic details are embedded in the two transport coefficients $D(\rho)$ and $\sigma(\rho)$ which appear in (1,2). We will compare our results with an exact solution of the symmetric exclusion process which corresponds to $D(\rho)=1$ and $\sigma(\rho)=2\rho(1-\rho)$ [6,22]. This is a system of diffusing particles on a lattice with simple exclusion interactions among them. Starting with microscopic dynamics we will derive an explicit formula of the two-time correlations $\langle \tau_i(t)\tau_j(t')\rangle_c$ and $\langle \mathcal{J}_i(t)\mathcal{J}_j(t')\rangle_c$ for the occupation number $\tau_i(t)=0,\ 1$ of the site i and for the particle current $\mathcal{J}_i(t)$ across a bond $(i,i+1)$. We show that, in the hydrodynamic limit, the formulas lead to their corresponding results for $\langle \rho(x,\tau)\rho(y,\tau')\rangle_c$ and $\langle J(x,\tau)J(y,\tau')\rangle_c$. These exact calculations allow us to directly verify the fluctuating hydrodynamics equation (1).

2. Macroscopic fluctuation theory

Starting with the fluctuating hydrodynamics equation (1) one can write the generating function

$$\left\langle e^{L\int_0^T d\tau \int_0^1 dx \lambda(x,\tau)J(x,\tau)}\right\rangle = \int \mathcal{D}[J,\rho]\delta\left(\partial_\tau\rho+\partial_x J\right) e^{L\int_0^T d\tau\int_0^1 dx\left[\lambda(x,\tau)J(x,\tau)-\frac{(J+D(\rho)\partial_x\rho)^2}{2\sigma[\rho]}\right]} \tag{4}$$

where T is an arbitrary large time. The path integral is over all density and current fields with a boundary condition

$$\rho(0,\tau)=\rho_a, \qquad \rho(1,\tau)=\rho_b. \tag{5}$$

The delta finction can be written as an integral over an additional field $\widehat{\rho}(x,\tau)$. This field $\widehat{\rho}(x,\tau)$ must vanish at the boundary as the particles are not conserved

$$\widehat{\rho}(0,\tau)=0=\widehat{\rho}(1,\tau). \tag{6}$$

Completing the Gaussian integration over the current $J(x,\tau)$ one arrives at

$$\left\langle e^{L\int_0^T d\tau \int_0^1 dx \lambda(x,\tau)J(x,\tau)}\right\rangle = \int \mathcal{D}[\widehat{\rho},\rho]\, e^{-L\,S[\widehat{\rho},\rho]}, \tag{7}$$

where $S[\widehat{\rho},\rho]$ can be written as a classical Action

$$S[\widehat{\rho},\rho]=\int_0^T d\tau\left(\int_0^1 dx\,\widehat{\rho}\,\partial_\tau\rho-\mathcal{H}[\widehat{\rho},\rho]\right) \tag{8}$$

with a Hamiltonian

$$\mathcal{H}[\widehat{\rho},\rho]=\int_0^1 dx\,\left(\frac{\sigma[\rho]}{2}\left(\partial_x\widehat{\rho}+\lambda\right)^2-D(\rho)\left(\partial_x\widehat{\rho}+\lambda\right)\partial_x\rho\right) \tag{9}$$

(Compare the expression with Action for large deviation of density.)

For a large system of length L the integral on the right had side in (7) is dominated by the minimal value of the Action. Denoting the paths associated to the least Action by $(\widehat{\rho},\rho)\equiv(p,q)$ one gets the cumulant generating function of current

$$\mu[\lambda]=\lim_{L\to\infty}L^{-1}\log\left\langle e^{L\int_0^T d\tau\int_0^1 dx\,\lambda(x,\tau)J(x,\tau)}\right\rangle=-\,S[p,q]. \tag{10}$$

The optimal fields satisfy Hamilton's equation

$$\partial_\tau p + \partial_x (\partial_x p + \lambda) = -\frac{\sigma'[q]}{2} (\partial_x p + \lambda)^2 \tag{11a}$$

$$\partial_\tau q - \partial_{xx} q = -\partial_x (\sigma[q](\partial_x p + \lambda)) \tag{11b}$$

The corresponding boundary conditions come from minimizing the Action and given by

$$p(x,T) = 0 \quad \text{and} \quad q(x,0) = \overline{\rho}(x) = \rho_a(1-x) + \rho_b x. \tag{12}$$

In addition there are conditions

$$p(0,\tau) = 0 = p(1,\tau), \quad \text{and} \quad q(0,\tau) = \rho_a, \quad q(1,\tau) = \rho_b \qquad \text{at all time } \tau. \tag{13}$$

The expression for the minimal Action can be simplified using the optimal equation (11b) leading to

$$\mu[\lambda] = -\int_0^T dt \int_0^1 dx \left\{ \frac{\sigma(q)}{2} (\partial_x p + \lambda)^2 + \lambda \partial_x q - \lambda \sigma(q) (\partial_x p + \lambda) \right\} \tag{14}$$

Remark: Using (10) and the definition

$$\mu[\lambda] = \int_0^T d\tau \int_0^1 dx \, \langle J(x,\tau) \rangle + \frac{1}{2} \int_0^T d\tau_1 d\tau_2 \int_0^1 dx_1 dx_2 \, \langle J(x_1,\tau_1) J(x_2,\tau_2) \rangle_c + \cdots \tag{15}$$

one gets for large L,

$$\langle J(x_1,t_1), \cdots, J(x_k,t_k) \rangle_c \simeq \frac{1}{L^{k-1}} c(x_1,t_1,\cdots,x_k,t_k), \tag{16}$$

where $c(x_1,t_1,\cdots,x_k,t_k)$ is a scaling function.

2.1. A perturbation solution

Introducing a parameter ϵ by defining

$$\lambda(x,t) = \epsilon \, h(x,t). \tag{17}$$

and expanding the optimal fields in powers of ϵ one writes

$$p = \epsilon p_1 + \epsilon^2 p_2 + \cdots \tag{18}$$

$$q = q_0 + \epsilon q_1 + \epsilon^2 q_2 + \cdots . \tag{19}$$

(Vanishing of the zeroth order term in p can be seen from the Hamilton's equation and the corresponding boundary condition.)

Substituting the expansion in (14) gives

$$\mu[\epsilon \, h] = \epsilon \mu_1[h] + \epsilon^2 \mu_2[h] + \cdots , \tag{20}$$

where

$$\mu_1[h] = -\int_0^{t_c} dt \int_0^1 dx \, h(x,t) \partial_x q_0(x,t), \tag{21}$$

and

$$\mu_2[h] = \int_0^{t_c} dt \int_0^1 dx \left(\sigma[q_0] h^2 - \frac{1}{2} \sigma[q_0] (\partial_x p_1)^2 - h \partial_x q_1 \right). \tag{22}$$

The multi-time cumulants are related to the terms in the above expansion of the cumulant generating function:

$$\mu_k[h] = \frac{1}{k!}\left[\prod_{\ell=1}^{k}\int_0^{t_c} dt_\ell \int_0^1 dx_\ell\, h(x_\ell,t_\ell)\right]\langle J(x_1,t_1)\cdots J(x_k,t_k)\rangle_c\,. \tag{23}$$

Using this definition for the linear order immediately leads us to the expected result

$$\langle J(x,t)\rangle = -\partial_x q_0(x,t). \tag{24}$$

For an explicit result, we solve the equation for $q_0(x,t)$ which is

$$\partial_t q_0 - \partial_{xx} q_0 = 0. \tag{25}$$

The corresponding boundary conditions are

$$q_0(x,0) = \rho_0(x), \qquad q_0(0,t) = \rho_a, \qquad \text{and} \qquad q_0(1,t) = \rho_b. \tag{26}$$

One can easily see that the solution is time independent and equal to the initial profile:

$$q_0(x,t) = \rho_0(x). \tag{27}$$

Substituting this result in the expression (24) we obtain

$$\langle J(x,t)\rangle = \rho_a - \rho_b. \tag{28}$$

Two-time correlation: The ϵ^2 order term μ_2 involves solution of the optimal fields upto the linear order in ϵ. We first solve for $p_1(x,t)$ which satisfies

$$\partial_t p_1 + \partial_{xx} p_1 = -\partial_x h \tag{29}$$

along with a boundary condition

$$p_1(x,t_c) = 0, \quad p_1(0,t) = 0, \quad \text{and} \quad p_1(1,t) = 0. \tag{30}$$

The corresponding solution can be written as

$$p_1(x,t) = -\int_t^{t_c} d\tau \int_0^1 dy\, \partial_y G(y,\tau|x,t) h(y,\tau) \tag{31}$$

where we defined

$$G(y,\tau|x,t) = 2\sum_{n=1}^{\infty} e^{-n^2\pi^2(\tau-t)}\sin(n\pi y)\sin(n\pi x) \quad \text{for all } \tau \geq t. \tag{32}$$

In deriving (31) we have used that $G(y,\tau|x,t) = 0$ when either y or x are at the boundary.

Similarly, we find that $q_1(x,t)$ satisfies an equation

$$\partial_t q_1 - \partial_{xx} q_1 = -\partial_x\left(\sigma[q_0]\left(\partial_x p_1 + h\right)\right) \tag{33}$$

with a the boundary condition

$$q_1(x,0) = 0, \quad \text{and} \quad q_1(0,t) = 0 = q_1(1,t). \tag{34}$$

The corresponding solution can also be written in terms of $G(y,\tau|x,t)$ as

$$q_1(x,t) = \int_0^t d\tau \int_0^1 dy\, \sigma[q_0(y,\tau)]\partial_y G(x,t|y,\tau)\left(\partial_y p(y,\tau) + h(y,\tau)\right). \tag{35}$$

Using the above solutions we determine the following integral which appears in the expression for $\mu_2[h]$ in (22).

$$\mathbb{I}_1 = \int_0^1 dt_1 \int_0^1 dx_1 h(x_1,t_1)\partial_{x_1} q_1(x_1,t_1). \tag{36}$$

Using (35) we rewrite the above as

$$\mathbb{I}_1 = \int_0^{t_c} dt_1 \int_0^{t_1} dt_2 \int_0^1 dx_1 \int_0^1 dx_2\, h(x_1,t_1)h(x_2,t_2)\,\sigma[\rho_0(x_2)]\partial_{x_1x_2}G(x_1,t_1|x_2,t_2)$$
$$+\int_0^{t_c} dt_1 \int_0^{t_1} dt_2 \int_0^1 dx_1 \int_0^1 dx_2 h(x_1,t_1)\,\sigma[\rho_0(x_2)]\partial_{x_1x_2}G(x_1,t_1|x_2,t_2)\partial_{x_2}p_1(x_2,t_2).$$

In writing the above equation we have used that $q_0[x,t]=\rho_0(x)$.

The second integral which appears in (22) is

$$\mathbb{I}_2 = \frac{1}{2}\int_0^{t_c} dt_2 \int_0^1 dx_2\,\sigma[\rho_0(x_2)]\left(\partial_{x_2}p_1\right)^2 \tag{37}$$

wching using (31) we rewrite as

$$\mathbb{I}_2 = -\frac{1}{2}\int_0^{t_c} dt_1 \int_0^{t_1} dt_2 \int_0^1 dx_1 \int_0^1 dx_2\, h(x_1,t_1)\sigma[\rho_0(x_2)]\partial_{x_1x_2}G(x_1,t_1|x_2,t_2)\partial_{x_2}p_1(x_2,t_2)$$

Combining altogether we obtain

$$\mathbb{I}_1+\mathbb{I}_2 = \int_0^{t_c} dt_1 \int_0^{t_1} dt_2 \int_0^1 dx_1 \int_0^1 dx_2\, h(x_1,t_1)h(x_2,t_2)\,\sigma[\rho_0(x_2)]\partial_{x_1x_2}G(x_1,t_1|x_2,t_2)$$
$$+\frac{1}{2}\int_0^{t_c} dt_1 \int_0^{t_1} dt_2 \int_0^1 dx_1 \int_0^1 dx_2 h(x_1,t_1)\,\sigma[\rho_0(x_2)]\partial_{x_1x_2}G(x_1,t_1|x_2,t_2)\partial_{x_2}p_1(x_2,t_2).$$

To further evaluate, we substitute the solution $p_1(x,t)$ from (31), and following a straightforward algebra we arrive at

$$\mathbb{I}_1+\mathbb{I}_2 = \frac{1}{2}\int_0^{t_c} dt_1 \int_0^{t_1} dt_2 \int_0^1 dx_1 \int_0^1 dx_2\, h(x_1,t_1)h(x_2,t_2)\Big\{\sigma[\rho_0(x_2)]\partial_{x_1x_2}$$
$$G(x_1,t_1|x_2,t_2) - \int_0^{t_2} d\tau \int_0^1 dy\,\sigma[\rho_0(y)]\partial_{x_1y}G(x_1,t_1|y,\tau)\partial_{x_2y}G(x_2,t_2|y,\tau)\Big\} + (1\leftrightarrow 2)$$

where the last term on the right hand side denotes the expression obtained by interchange of indices 1 and 2 in the terms precceeding it.

Using the above result in (22) along with the definition of the second cumulant in (23) we arrive at

$$\langle J(x_1,t_1)J(x_2,t_2)\rangle_c = \sigma[\rho_0(x_1)]\delta(x_1-x_2)\delta(t_1-t_2)$$
$$+f(x_1,t_1,x_2,t_2)\Theta(t_2-t_1)+f(x_2,t_2,x_1,t_1)\Theta(t_1-t_2) \tag{38}$$

where

$$f(x_1,t_1,x_2,t_2) = -\sigma[\rho_0(x_1)]\partial_{x_2x_1}G(x_2,t_2|x_1,t_1)$$
$$+\int_0^{t_1} d\tau \int_0^1 dy\,\sigma[\rho_0(y)]\partial_{x_2y}G(x_2,t_2|y,\tau)\partial_{x_1y}G(x_1,t_1|y,\tau). \tag{39}$$

This leads us to the scaling function in (10):

$$c(x_1,t_1;x_2,t_2) = \sigma[\overline{\rho}(x_1)]\delta(x_1-x_2)+f(x_1,t_1,x_2,t_2)\Theta(t_2-t_1)+f(x_2,t_2,x_1,t_1)\Theta(t_1-t_2) \tag{40}$$

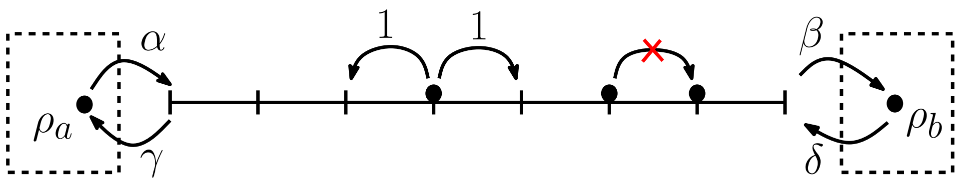

Figure 1. Jump rates in the symmetric exlusion process coupled with a reservoir of density ρ_a at the left boundary and a reservoir of density ρ_b at the right boundary.

3. The symmetric exclusion process using microscopic dynamics

The symmetric exclusion process is defined on a one-dimensional lattice of L sites, coupled with two reservoirs of density ρ_a and ρ_b. Particles within the bulk jump from a site to one of its neighbor sites as long as the jump respects simple exclusion: at any time there could be at most one particle at a site. The time scale is set by choosing bulk jump rates equal to 1. (This is equivalent to setting $D(\rho) = 1$ in the hydrodynamic limit.) At the boundary sites, the coupling to the reservoirs is modeled by injection and extraction rates of particles as shown in figure 1; the density of the reservoirs are related [6] to these boundary rates by

$$\rho_a = \frac{\alpha}{\alpha+\gamma}; \qquad \rho_b = \frac{\delta}{\beta+\delta}$$

At long time, the system reaches a steady state where the average occupation per site is linear in space, and the average current is constant [6, 8, 41],

$$\langle \tau_i \rangle = \frac{1}{N}\left[\rho_a\,(L+b-i) + \rho_b\,(i-1+a)\right]; \qquad \langle \mathcal{J}_i \rangle \equiv \langle \mathcal{J} \rangle = \frac{(\rho_a - \rho_b)}{N} \tag{41}$$

where we defined

$$a = \frac{1}{\alpha+\gamma}; \qquad b = \frac{1}{\beta+\delta}; \qquad \text{and} \quad N = L + a + b - 1 \tag{42}$$

The steady state two-point correlation function of occupation variables as well as higher order correlation functions are known [6, 8, 41]. For example,

$$\langle \tau_i \tau_j \rangle_c = \begin{cases} -\left[\dfrac{(\rho_a-\rho_b)^2}{N^2(N-1)}\right](i+a-1)(L+b-j) & \text{for } i<j \\ \langle \tau_i \rangle\,(1 - \langle \tau_i \rangle) & \text{for } i=j \end{cases} \tag{43}$$

Using the definition of the model, one can write the time evolution of these averages. For example,

$$\frac{d\,\langle \boldsymbol{\tau}(t) \rangle}{dt} = -\boldsymbol{M} \cdot \langle \boldsymbol{\tau}(t) \rangle + \mathbf{B} \tag{44}$$

where

$$\boldsymbol{\tau}(t) = \begin{pmatrix} \tau_1(t) \\ \tau_2(t) \\ . \\ . \\ . \\ \tau_L(t) \end{pmatrix} \; ; \; \mathbf{M} = \begin{pmatrix} 1+\frac{1}{a} & -1 & 0 & . & . & . & . \\ -1 & 2 & -1 & 0 & . & . & . \\ 0 & -1 & 2 & -1 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & 0 & -1 & 2 & -1 \\ . & . & . & . & 0 & -1 & 1+\frac{1}{b} \end{pmatrix} \; ; \; \mathbf{B} = \begin{pmatrix} \alpha \\ 0 \\ . \\ 0 \\ \delta \end{pmatrix} \tag{45}$$

We shall use bold font to denote matrices.

The solution of (51) is simple to write in terms of a microscopic Green's function $\mathbf{g}(t) = [g_{i,j}(t)]_{L\times L}$ defined as the solution of

$$\frac{d\mathbf{g}(t)}{dt} + \boldsymbol{M} \cdot \mathbf{g}(t) = \delta(t)\mathbf{1} \tag{46}$$

where $\mathbf{1}$ is the identity matrix. The solution of (51) in the large time limit gives steady state density

$$\langle \boldsymbol{\tau} \rangle = \int_0^\infty dt \; \mathbf{g}(t) \cdot \mathbf{B} \tag{47}$$

The average current in the steady state is expressed in terms of the average density

$$\langle \mathcal{J}_i \rangle = \begin{cases} \alpha - \frac{1}{a} \langle \tau_1 \rangle & \text{for } i = 0 \\ \langle \tau_i \rangle - \langle \tau_{i+1} \rangle & \text{for } 1 \le i < L \\ \frac{1}{b} \langle \tau_L \rangle - \delta & \text{for } i = L \end{cases} \tag{48}$$

Two-time correlations of occupation variables are derived in [Sadhu, Derrida, JSTAT]

3.1. Correlation of current

We present the result first and defer the proof to the **??**. In the steady state

$$\langle \mathcal{J}_i(t) \mathcal{J}_j(0) \rangle_c = \langle W_i(t) \rangle_1 \langle U_j \rangle - \langle W_i(t) \rangle_2 \langle V_j \rangle \tag{49}$$

where

$$\boldsymbol{W}(t) = \begin{pmatrix} -\frac{1}{a} n_1(t) \\ . \\ . \\ n_i(t) - n_{i+1}(t) \\ . \\ . \\ \frac{1}{b} n_L(t) \end{pmatrix} \; ; \quad \boldsymbol{U} = \begin{pmatrix} \alpha(1-\tau_1) \\ . \\ . \\ \tau_i(1-\tau_{i+1}) \\ . \\ . \\ \beta\tau_L \end{pmatrix} \; ; \quad \boldsymbol{V} = \begin{pmatrix} \gamma\tau_1 \\ . \\ . \\ (1-\tau_i)\tau_{i+1} \\ . \\ . \\ \delta(1-\tau_L) \end{pmatrix} \tag{50}$$

where the column vectors are of size $i = 0, ..., L$.

The $\langle n_i(t) \rangle_{1(2)}$ are the solution of

$$\frac{\langle \boldsymbol{n}(t) \rangle_{1(2)}}{dt} = -\boldsymbol{M} \cdot \langle \boldsymbol{n}(t) \rangle_{1(2)} \tag{51}$$

with $\boldsymbol{M}$ given in (45) and initial condition

$$\langle n_i(0)\rangle_1 = \begin{cases} -\langle\tau_j\rangle & \text{for } i=j \\ 1-\langle\tau_{j+1}\rangle & \text{for } i=j+1 \\ \dfrac{\langle\tau_i\tau_j\rangle_c - \langle[\tau_i-\langle\tau_i\rangle]\,\tau_j\tau_{j+1}\rangle}{\langle\tau_j(1-\tau_{j+1})\rangle} & \text{for the rest.} \end{cases} \tag{52}$$

$$\langle n_i(0)\rangle_2 = \begin{cases} 1-\langle\tau_j\rangle & \text{for } i=j \\ -\langle\tau_{j+1}\rangle & \text{for } i=j+1 \\ \dfrac{\langle\tau_i\tau_{j+1}\rangle_c - \langle[\tau_i-\langle\tau_i\rangle]\,\tau_j\tau_{j+1}\rangle}{\langle(1-\tau_j)\tau_{j+1}\rangle} & \text{for the rest} \end{cases} \tag{53}$$

with a choice that $\langle\tau_0\rangle=\rho_a$ and $\langle\tau_{L+1}\rangle = 1-\rho_b$

3.2. An explicit solution

Solution of (51) is

$$\langle\boldsymbol{n}_i(t)\rangle_{1(2)} = \sum_\lambda e^{-\lambda\,t}\psi_i(\lambda)\left[\sum_\ell \psi_\ell^\star(\lambda)\,\langle n_\ell(0)\rangle_{1(2)}\right] \tag{54}$$

where $\psi_i(\lambda)$ is normalized eigen function of $\boldsymbol{M}$ corresponding to eigenvalue λ, and $\star$ denotes complex conjugate. Substituting this solution in (49) one gets

$$\langle\mathcal{J}_i(t)\mathcal{J}_j(0)\rangle_c = \begin{cases} \sum_\lambda e^{-\lambda\,s}\left[-\dfrac{\psi_1(\lambda)}{a}\right]\sum_\ell \psi_\ell(\lambda)\,A_{\ell,j} & \text{for } i=0 \\ \sum_\lambda e^{-\lambda\,s}\left[\psi_i(\lambda)-\psi_{i+1}(\lambda)\right]\sum_\ell \psi_\ell(\lambda)\,A_{\ell,j} & \text{for rest} \\ \sum_\lambda e^{-\lambda\,s}\left[\dfrac{\psi_L(\lambda)}{b}\right]\sum_\ell \psi_\ell(\lambda)\,A_{\ell,j} & \text{for } i=L \end{cases} \tag{55}$$

where

$$A_{\ell,j} = \langle n_\ell(0)\rangle_1\langle U_j\rangle - \langle n_\ell(0)\rangle_2\langle V_j\rangle \tag{56}$$

Substituting (52), (53) and (50) one gets

$$A_{\ell,j} = \begin{cases} -\dfrac{1}{a}\langle\tau_\ell\tau_1\rangle_c + \delta_{\ell,1}\left[\alpha-(\alpha-\gamma)\langle\tau_1\rangle\right] & \text{for j=0} \\ \langle\tau_\ell\tau_j\rangle_c - \langle\tau_\ell\tau_{j+1}\rangle_c + (\delta_{\ell,j+1}-\delta_{\ell,j})\left\langle(\tau_j-\tau_{j+1})^2\right\rangle & \text{for rest} \\ \dfrac{1}{b}\langle\tau_\ell\tau_L\rangle_c + \delta_{\ell,L}\left[(\delta-\beta)\langle\tau_L\rangle-\delta\right] & \text{for j=0} \end{cases} \tag{57}$$

It is straighforward to show that the eigenfunctions are

$$\psi_j(\lambda) = z_\lambda^j - z_\lambda^{-j+1}f_a(z) \tag{58}$$

where we defined

$$f_r(z_\lambda) = \frac{1+\left(\frac{1}{r}-1\right)z_\lambda}{\frac{1}{r}-1+z_\lambda}. \tag{59}$$

The z_λ are the complex roots of the polynomial equation

$$z_\lambda^{2L} = f_a(z_\lambda) f_b(z_\lambda) \tag{60}$$

The eigenvalues are determined in terms of these roots as

$$\lambda = 2 - z_\lambda - z_\lambda^{-1}. \tag{61}$$

There are L distinct roots which lead to L linearly independent eigenfunctions and associated distinct eigenvalues.

To apply the results in the formula (55) we write

$$\psi_j(\lambda) - \psi_{j+1}(\lambda) = (1 - z_\lambda)\left(z_\lambda^j + z_\lambda^{-j} f_a(z_\lambda)\right), \tag{62}$$

which can be shown from (58). Similarly we get

$$\sum_k \psi_k(\lambda)^2 = \frac{z_\lambda^2}{1 - z_\lambda^2}\left[1 - f_a(z_\lambda) f_b(z_\lambda)\right]\left[1 + \frac{f_a(z_\lambda)}{f_b(z_\lambda)}\right] - 2L z_\lambda f_a(z_\lambda) \tag{63}$$

where in addition to (58) we used (60).

Substituting the above two expressions in the solution (55) we arrive at

$$\langle \mathcal{J}_j(s) \mathcal{J}_i(0)\rangle_c = \sum_\lambda e^{-\lambda s} \frac{(1 - z_\lambda^2)\, H_j(z_\lambda) \sum_\ell \left(z_\lambda^\ell - z_\lambda^{-\ell+1} f_a(z_\lambda)\right) R_{\ell,i}}{z_\lambda^2 \left[1 - f_a(z_\lambda) f_b(z_\lambda)\right]\left[1 + \frac{f_a(z_\lambda)}{f_b(z_\lambda)}\right] - 2L z_\lambda (1 - z_\lambda^2) f_a(z_\lambda)} \tag{64}$$

where

$$H_j(z_\lambda) = \left[1 + (b-1)\delta_{j,L}\right]^{-1}\left\{(1 - z_\lambda)\left[z_\lambda^j + z_\lambda^{-j} f_a(z_\lambda)\right] + z_\lambda^L \left[z_\lambda - \frac{1}{f_b(z_\lambda)}\right]\delta_{j,L}\right\} \tag{65}$$

where $\delta_{\ell,j}$ is the Kronecker delta.

The remaining quantity to evaluate is $R_{\ell,i}$. Using the initial conditions (52)-(53) in the formula (**??**) we find that $R_{\ell,i}$ involves up to two-point correlations $\langle \tau_j \tau_k \rangle_{ss}$. We use the known results of the correlations [6] and arrive at

$$R_{\ell,i} = \begin{cases} \Omega_{\ell,i} + \kappa_i \left[\delta_{\ell,i+1} - \delta_{\ell,i}\right] + \Gamma_i \left[\delta_{\ell,i+1} + \delta_{\ell,i}\right] + \langle \mathcal{J}\rangle_{ss}\left[1 - 2\langle \tau_i \rangle_{ss}\right]\delta_{\ell,i} & \text{for } 1 \le i < L \\ \Omega_{\ell,L} + \left[\frac{\langle \mathcal{J}\rangle_{ss}^2 N}{(N-1)}(1-b) - \delta(1 - b\delta)\right]\delta_{\ell,L} & \text{for } i = L \end{cases} \tag{66}$$

where we defined

$$\kappa_i = \langle \tau_i \rangle_{ss}\left[1 - \langle \tau_i \rangle_{ss}\right] + \frac{\langle \mathcal{J}\rangle_{ss}^2}{N-1}(a + i - 1)(b + L - i - 1) \tag{67}$$

$$\Gamma_i = \frac{\langle \mathcal{J}\rangle_{ss}^2}{N-1}(a + i - 1) \tag{68}$$

and

$$\Omega_{\ell,i} = \begin{cases} -\frac{\langle \mathcal{J}\rangle_{ss}^2}{(N-1)}(a + \ell - 1) & \text{for } \ell \le i \\ \frac{\langle \mathcal{J}\rangle_{ss}^2}{(N-1)}(b + L - \ell) & \text{for } \ell \ge i + 1. \end{cases} \tag{69}$$

We deferred a detailed derivation of the expression for $R_{\ell,i}$ in the **??**.

The above constitutes the solution for the auto-correlation of current. In the rest we analyse the result and derive a closed form expression.

Appendix A. Derivation of the correlation of current.

Current $\mathcal{J}_i(t) = \frac{Y_i(t)}{dt}$ where $Y_i(t)$ is the number of jumps across bond $(i, i+1)$ in a small time window $[t - dt, t]$. Then

$$\langle \mathcal{J}_i(t)\mathcal{J}_j(0)\rangle = \lim_{dt\to 0} \frac{\langle Y_i(t)Y_j(0)\rangle}{dt^2}$$

In the steady state one can rewrite the above formula as

$$\langle \mathcal{J}_i(t)\mathcal{J}_j(0)\rangle = \langle \mathcal{J}_i(t)\rangle_1 \frac{P[Y_j(0)=1]}{dt} - \langle \mathcal{J}_i(t)\rangle_2 \frac{P[Y_j(0)=-1]}{dt} \tag{A.1}$$

for $t \geq 0$, where $P[Y_j(0)]$ denotes probability of $Y_j(0)$ jumps in a small time window $[-dt, 0]$. The $\langle\rangle_{1(2)}$ denotes average conditioned on $Y_j(0) = 1\ (-1)$.

In the steady state

(a) for $i = 0$

$$P(Y_0=1)=\alpha\,[1-\langle\tau_1\rangle]\,dt; \qquad P(Y_0=-1)=\gamma\,\langle\tau_1\rangle\,dt \tag{A.2}$$

(b) for $1 \leq i < L$

$$P(Y_i=1)=\langle\tau_i\,(1-\tau_{i+1})\rangle\ dt; \qquad P(Y_i=-1)=\langle(1-\tau_i)\,\tau_{i+1}\rangle\ dt \tag{A.3}$$

(c) for $i = L$

$$P(Y_L=1)=\beta\,\langle\tau_L\rangle\ dt; \qquad P(Y_L=-1)=\delta\,(1-\langle\tau_L\rangle)\ dt \tag{A.4}$$

The conditional averages

$$\langle \mathcal{J}_i(t)\rangle_{1(2)} = \begin{cases} \alpha - \dfrac{1}{a}\,\langle\tau_1(t)\rangle_{1(2)} & \text{for } i = 0 \\ \langle\tau_i(t) - \tau_{i+1}(t)\rangle_{1(2)} & \text{for } 1 \leq i < L \\ \dfrac{1}{b}\,\langle\tau_L(t)\rangle_{1(2)} - \delta & \text{for } i = L \end{cases} \tag{A.5}$$

The conditioned average satisfies the same equation (51); the difference is only in the initial condition:

(a) for $i = j$,

$$\langle\tau_j(0)\rangle_1 = 0 \qquad \text{and} \qquad \langle\tau_j(0)\rangle_2 = 1 \tag{A.6}$$

(b) for $i = j + 1$,

$$\langle\tau_{j+1}(0)\rangle_1 = 1 \qquad \text{and} \qquad \langle\tau_{j+1}(0)\rangle_2 = 0 \tag{A.7}$$

This condition (A.6, A.7) comes from $Y_i(0) = 1\ (-1)$ in the time window $[-dt, 0]$. For the rest of the sites,

(c) for $i \neq j$ or $j + 1$, and

(i) for bulk sites $1 \leq j \leq L - 1$

$$\langle\tau_i(0)\rangle_1 = \frac{\langle\tau_i\tau_j(1-\tau_{j+1})\rangle}{\langle\tau_j(1-\tau_{j+1})\rangle} \qquad \text{and} \qquad \langle\tau_i(0)\rangle_2 = \frac{\langle\tau_i(1-\tau_j)\tau_{j+1}\rangle}{\langle(1-\tau_j)\tau_{j+1}\rangle} \tag{A.8}$$

(ii) for $j=0$

$$\langle\tau_i(0)\rangle_1 = \frac{\langle\tau_i(1-\tau_1)\rangle}{1-\langle\tau_1\rangle} \qquad \text{and} \qquad \langle\tau_i(0)\rangle_2 = \frac{\langle\tau_i\tau_1\rangle}{\langle\tau_1\rangle} \tag{A.9}$$

(iii) for $j=L$

$$\langle\tau_i(0)\rangle_1 = \frac{\langle\tau_i\tau_L\rangle}{\langle\tau_L\rangle} \qquad \text{and} \qquad \langle\tau_i(0)\rangle_2 = \frac{\langle\tau_i(1-\tau_L)\rangle}{(1-\langle\tau_L\rangle)} \tag{A.10}$$

Condition (A.8-A.10) is obtained using that $\tau_i = 0$ or 1, and

$$\langle\tau_i\rangle_{1(2)} = P\left[\tau_i = 1 | Y_j = 1\,(-1)\right] = \frac{P\left[\tau_i = 1, Y_j = 1\,(-1)\right]}{P\left[Y_j = 1\,(-1)\right]} \tag{A.11}$$

where $P\left[\tau_i, Y_j\right]$ is the joint probability and $P\left[\tau_i|Y_j\right]$ is the conditional probability of τ_i given Y_j in the steady state.

Substituting the above results in (A.1) and expressing in terms of $n_j(t) = \tau_j(t) - \langle\tau_j\rangle$ one gets

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