# Multi-time correlation of integrated current in diffusive system on infinite line

**Abstract.** We study dynamical properties of integrated current in one dimensional diffusive system using macroscopic fluctuations theory. For the simplest example of diffusing point particles with hard-core repulsion we find an exact expression of the cumulant generating functional of history of current. This allows one to extract all multi-time correlations of integrated current. We verify our results from an exact calculation starting from microscopic rates (still to finish). For symmetric exclusion process we derive an exact result of two-time correlation.

*Keywords*:.

## 1. Hydrodynamic formulation

The starting point is a fluctuating hydrodynamic equation of the macroscopic density profile $\rho(x,t)$ of the interacting diffusing particles.

$$\partial_t \rho = \partial_x \left[ D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right], \tag{1}$$

where $\eta(x,t)$ is a Gaussian noise with mean zero and covariance

$$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t'). \tag{2}$$

The angular bracket denotes ensemble average.

We consider the history of the integrated current $Q(t)$ at the origin in a time window $[0,T]$. The integrated current at a time $t$ is defined in terms of the density profile by

$$Q(t) = \int_0^\infty dx\, [\rho(x,t) - \rho(x,0)] \tag{3}$$

The statistics of the history of $Q(t)$ can be characterized in terms of the generating function $\langle \exp[\int dt\, \lambda(t) Q(t)]\rangle$, where $\lambda(t)$ is a continuous function. The logarithm of the average is the cumulant generating functional

$$\mu[\lambda] = \log \left[ \left\langle \exp \left( \int_0^T dt\, \lambda(t) Q(t) \right) \right\rangle \right], \tag{4}$$

where we use square brackets to denote a functional.

Starting with the fluctuating hydrodynamic equation (1) for the density profile $\rho(x,t)$, one writes the moment generating function as a path integral

$$\langle e^{\int_0^T dt \lambda(t) Q(t)} \rangle = \int \mathcal{D}\,[\rho, \hat\rho]\, e^{-S[\rho,\hat\rho]}, \tag{5}$$

with the action

$$S[\rho,\hat\rho] = -\int_0^T dt\lambda(t)Q_t[\rho] + F[\rho(x,0)] \\ + \int_0^T dt \int_{-\infty}^{\infty} dx \left\{ \hat\rho\partial_t\rho - \frac{\sigma(\rho)}{2}\left(\partial_x\hat\rho\right)^2 + D(\rho)\left(\partial_x\rho\right)\left(\partial_x\hat\rho\right)\right\}.$$

The function $F[\rho]$ characterizes the initial state: for the quenched setting $F[\rho] = 0$ whereas for the annealed setting it is the free energy associated to the initial density fluctuation $\rho(x)$. For the step initial state with uniform average density

$$\rho_0(x) = \Theta(-x)\rho_a + \Theta(x)\rho_b \tag{6}$$

this is

$$F\left[\rho(x)\right] = \int_{-\infty}^{\infty} dx \int_{\rho_0(x)}^{\rho(x)} dr \frac{2D(r)}{\sigma(r)}\left[\rho(x) - r\right] \tag{7}$$

The action $S[\rho,\hat\rho]$ grows with increasing time and at large $T$ the path integral is dominated by the paths that corresponds to the least action. We denote this optimal path by $(q,p) = (\rho,\hat\rho)$. One finds that the least action path

$$\partial_t q - \partial_x\left(D(q)\partial_x q\right) = -\partial_x\left(\sigma(q)\partial_x p\right), \tag{8}$$

$$\partial_t p + D(q)\partial_{xx}p = -\frac{1}{2}\sigma'(q)\left(\partial_x p\right)^2 - \lambda(t)\Theta(x). \tag{9}$$

The boundary conditions also come from minimizing action: for annealed case

$$p(x,0) = \Theta(x)\int_0^T dt\ \lambda(t) + \int_{\rho_0(x)}^{q(x,0)} dr \frac{2D(r)}{\sigma(r)} \tag{10}$$

$$p(x,T) = 0. \tag{11}$$

Here $\Theta(x)$ is the Heaviside step function. For quenched case

$$q(x,0) = \rho_0(x), \tag{12}$$

$$p(x,T) = 0. \tag{13}$$

Using the least action equations one can simplify the expression for the cumulant generatng function leading to

$$\mu[\lambda] = \int_0^T dt\lambda(t)Q_t[q] - F[q(x,0)] - \int_0^T dt\int_{\infty}^{\infty} dx\,\frac{\sigma(q(x,t))}{2}\left(\partial_x p(x,t)\right)^2, \tag{14}$$

with the corresponding $F$ for the annealed or quenched setting. In establishing (14) we have taken into account that the derivatives $\partial_x p$ and $\partial_x q$ vanish as $x \to \pm\infty$.

As a self consistency check we verify that by substituting $\lambda(t) = B\delta(T-t)$, we get the variational problem for the analysis of the current at final time, derived in [Derrida, Gerschenfeld].

## 2. Analysis for $\sigma(\rho) = 2\rho$ and $D(\rho) = 1$

This corresponds to non-interacting particles or particles with hard-core repulsive interaction. The cumulant generating function

$$\mu[\lambda(t)] = \int_0^T dt\ \lambda(t)Q(t) - F[q(x,0)] - \int_{-\infty}^{\infty} dx \int_0^T dt\ q\left(\partial_x p\right)^2 . \tag{15}$$

where

$$F[q] = \begin{cases} \displaystyle\int_{-\infty}^{\infty} dx q(x,0) \ln\left[\frac{q(x,0)}{\rho_0(x)}\right] - \int_{-\infty}^{\infty} dx\left[q(x,0) - \rho_0(x)\right], & \text{for annealed} \\ 0. & \text{for quenched} \end{cases} \tag{16}$$

The optimal equations determining the $q(x,t)$ and $p(x,t)$ are

$$\partial_t q - \partial_{xx} q = -\partial_x\left(2q\partial_x p\right), \tag{17}$$

$$\partial_t p + \partial_{xx} p = -\left(\partial_x p\right)^2 - \lambda(t)\Theta(x). \tag{18}$$

The boundary conditions are

$$p(x,T) = 0 \tag{19}$$

for both annealed and quenched setting whereas

$$q(x,0) = \begin{cases} \rho_0(x)\, e^{p(x,0)-\Theta(x)\int_0^T dt\ \ \lambda(t)} & \text{for annealed ,} \\ \rho_0(x) & \text{for quenched.} \end{cases} \tag{20}$$

Before we procced with an explicit solution of the optimal equations, we use them to simplify the expression for the cumulant generating function. For this we use the identity

$$q(\partial_x p)^2 = \partial_t(pq) - \partial_x\left[p\partial_x q - q\partial_x p - 2qp\partial_x p\right] + \lambda(t)\Theta(x)q, \tag{21}$$

which can be proved using the optimal equations. Substituting this in the expression for $\mu$ in (15) and using the boundary conditions we find

$$\mu[\lambda] = \begin{cases} \displaystyle\int_{-\infty}^{\infty} dx\left[q(x,0) - \rho_0(x)\right] & \text{for annealed,} \\ \displaystyle\int_{-\infty}^{\infty} dx\left\{p(x,0) - \Theta(x)\int_0^T dt\lambda(t)\right\}\rho_0(x) & \text{for quenched.} \end{cases} \tag{22}$$

In order to solve the optimal equation we make a change of variables $p = \ln P_\lambda$ and $q = R_\lambda P_\lambda$. This yields the corresponding equations

$$\partial_t P_\lambda + \partial_{xx} P_\lambda = -\lambda(t)\Theta(x)P_\lambda \qquad \text{and} \qquad \partial_t R_\lambda - \partial_{xx} R_\lambda = \lambda(t)\Theta(x)R_\lambda.$$

The boundary conditions in terms of this new variables yield $P_\lambda(x,T) = 1$ and

$$R_\lambda(x,0) = \begin{cases} \rho_0(x)\, e^{-\Theta(x)\int_0^T dt\lambda(t)} & \text{for annealed,} \\ \dfrac{\rho_0(x)}{P_\lambda(x,0)} & \text{for quenched.} \end{cases} \tag{23}$$

Using the above boundary condition, the cumulant generating function can be expressed in terms of these new variables. For the annealed case we find

$$\mu_{\mathcal{A}}[\lambda] = \int_{-\infty}^{\infty} dx\, \rho_0(x) \left[ P_\lambda(x,0) e^{-\Theta(x)\int_0^T dt \lambda(t)} - 1 \right] \tag{24}$$

whereas for quenched this is

$$\mu_{\mathcal{Q}}[\lambda] = \int_{-\infty}^{\infty} dx\, \rho_0(x) \ln \left[ P_\lambda(x,0) e^{-\Theta(x)\int_0^T dt \lambda(t)} \right]. \tag{25}$$

We only need the solution for $P_\lambda(x,0)$ in order to determine the generating function.

The expression could be rewritten in an alternate form using a symmetry

$$P_\lambda(-x,t) = P_{-\lambda}(x,t) e^{\int_t^T dt' \lambda(t')} \tag{26}$$

which can be derived using the equation for $P$. This leads to

$$\mu_{\mathcal{A}}[\lambda] = \rho_b \int_0^{\infty} dx \left[ P_\lambda(x,0) e^{-\int_0^T dt \lambda(t)} - 1 \right] + \rho_a \int_0^{\infty} dx \left[ P_{-\lambda}(x,0) e^{\int_0^T dt \lambda(t)} - 1 \right] \tag{27}$$

$$\mu_{\mathcal{Q}}[\lambda] = \rho_b \int_0^{\infty} dx \ln \left[ P_\lambda(x,0) e^{-\int_0^T dt \lambda(t)} \right] + \rho_a \int_0^{\infty} dx \ln \left[ P_{-\lambda}(x,0) e^{\int_0^T dt \lambda(t)} \right]. \tag{28}$$

In this form one can easily see that for $\rho_a = \rho_b$ the cumulant generating function is even function of $\lambda$.

The solution for $P_\lambda(x,t)$ can be expressed in the form of an integral equation

$$P_\lambda(x,t) = 1 + \int_t^T dt_1 \int_0^{\infty} dz_1 \lambda(t_1) g(z_1, t_1|x,t) P_\lambda(z_1,t_1), \tag{29}$$

where $g$ is the diffusion propagator

$$g(z,\tau|x,t) = \frac{1}{\sqrt{4\pi(\tau - t)}} \exp\left[ -\frac{(z-x)^2}{4(\tau - t)} \right] \tag{30}$$

At this stage we introduce a parameter $\epsilon$ by defining

$$\lambda(t) = \epsilon\, h(t). \tag{31}$$

An iterative expansion transforms the solution for $P_\lambda$ as

$$P_\lambda(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n \int_t^T dt_1 \int_{t_1}^T dt_2 \cdots \int_{t_{n-1}}^T dt_n h(t_1) \cdots h(t_n)\; K_n[x,t,t_1,\cdots,t_n], \tag{32}$$

where we defined

$$K_n[x,t,t_1,\cdots,t_n] = \int_0^{\infty} dz_1 \cdots \int_0^{\infty} dz_n\; g(z_n,t_n|z_{n-1},t_{n-1}) \cdots\; g(z_1,t_1|x,t). \tag{33}$$

The above solution substituted in the expressions (27)-(28) yields the result for the cumulant generating functions.

Using the results one can show that for annealed case the $n$ point cumulant with $t_1 \le t_2 \le \cdots \le t_n$ is

$$\langle Q(t_1), \ldots, Q(t_n) \rangle_{annealed} = (\rho_a + (-1)^n \rho_b) \sum_{k=1}^{n} \sum_{\sigma_k} (-1)^k \Psi_k(\sigma_k \mathbf{t}) \tag{34}$$

where the $\sigma_k\mathbf{t}$ is a subset of $k$ elements chosen from the set $\{t_1,\dots,t_n\}$ and ordering them according to their increasing values. For example $\sigma_{n-1}\mathbf{t}$ includes elements

$$\{(t_1,\dots,t_{n-1}),(t_2,\dots,t_n),(t_1,t_3,\dots,t_{n-1}),\dots,(t_1,\dots,t_{n-2},t_n))\}. \tag{35}$$

The quantity $\Psi_k$ is defined as

$$\Psi_n(t_1,\dots,t_n)=\int_0^\infty dx[K_n(x,0,t_1,\dots,t_n)-1] \tag{36}$$

One can verify that this leads to first few cumulants

$$\langle Q(t)\rangle \qquad = (\rho_a-\rho_b)\frac{1}{\sqrt{t}} \tag{37}$$

$$\langle Q(t_1)Q(t_2)\rangle = (\rho_a+\rho_b)\frac{1}{2\sqrt{\pi}}[\sqrt{t_1}+\sqrt{t_2}-\sqrt{|t_2-t_1|}] \tag{38}$$

Similar computation can be done for the annealed case (I still have to finish this)

*Microscopic analysis* *** Can one prove the result by counting ? ***

## 3. Two-time correlation

We consider the equilibrium case with $\rho_a=\rho_b=\rho$. In this case the average current $\langle Q(t)\rangle=0$. We write a series expansion

$$q(x,t)=\rho+\epsilon\, q_1(x,t)+\epsilon^2\, q_2(x,t)+\cdots, \tag{39}$$

$$p(x,t)=\epsilon\, p_1(x,t)+\epsilon^2\, p_2(x,t)+\cdots, \tag{40}$$

where we have used that, for $\epsilon=0$ the solution $q(x,t)=\rho$ and $p(x,t)=0$.

The cumulant generating function can also be expanded in a series as

$$\mu[\epsilon h(t)]=\epsilon\mu_1[h(t)]+\epsilon^2\mu_2[h(t)]+\cdots. \tag{41}$$

By definition, the coefficient of $\epsilon^n$ gives the $n$-time cumulant of the $X(t)$.

$$\mu_n[h]=\int_0^T dt_1 ... \int_0^T dt_n\; h(t_1)...h(t_n)\;\frac{1}{n!}\langle X(t_1)...X(t_n)\rangle_c. \tag{42}$$

With the series expansion (39)–(40), the optimal equations at the linear order becomes

$$\partial_t p_1 + D(\rho)\partial_{xx}p_1 = -h(t)\Theta(x), \tag{43}$$

$$\partial_t q_1 - D(\rho)\partial_{xx}q_1 = -\sigma(\rho)\partial_{xx}p_1., \tag{44}$$

One can similarly find the linear order term of current $Q(t)$ is

$$Q_1(t)=\int_0^\infty dx\,[q_1(x,t)-q_1(x,0)]\,, \tag{45}$$

Substituting the series expansion in the expression (14) we get

$$\mu_2[h(t)]=\int_0^T dt\int_0^\infty dx\; h(t)\,[q_1(x,t)-q_1(x,0)]-F_2-\frac{\sigma(\rho)}{2}\int_0^T dt\int_{-\infty}^\infty dx\,(\partial_x p_1)^2 \tag{46}$$

where $F_2$ is the order $\epsilon^2$ term of the $F[q(x,0)]$

$$F_2 = \begin{cases} 0 & \text{for quenched} \\ \dfrac{D(\rho)}{\sigma(\rho)} \displaystyle\int_{-\infty}^{\infty} dx\,(q_1(x,0))^2 & \text{for annealed} \end{cases} \tag{47}$$

Comparing with the analysis in [Krapivsky, Mallick, Sadhu, JSM 2015] one can easily see that

$$\langle Q(t_1)Q(t_2)\rangle_{quench} = \frac{\sigma(\rho)}{\sqrt{\pi D(\rho)}}\frac{1}{2}\left[\sqrt{t_1+t_2} - \sqrt{|t_1-t_2|}\right]. \tag{48}$$

whereas

$$\langle Q(t_1)Q(t_2)\rangle_{anneal} = \frac{\sigma(\rho)}{\sqrt{\pi D(\rho)}}\frac{1}{2}\left[\sqrt{t_1} + \sqrt{t_2} - \sqrt{|t_1-t_2|}\right] \tag{49}$$

*Non-equilibrium case*

The perturbative analysis can be done for $D(\rho) = 1$.

## 4. Microscopic calculation for symmetric exclusion process

Consider a symmetric exclusion process on an infinite line with step initial condition $\rho_0(x) = \rho_a\Theta(-x) + \rho_b\Theta(x)$. We derive the two point correlation of integrated current between any two sites.

We start with writing the equation for average occupation variable

$$\frac{\langle\tau(t)\rangle}{dt} = M\langle\tau(t)\rangle \tag{50}$$

where $\tau = \{\cdots, \tau_i, \cdots\}$ and $M$ is the Laplacian matrix

$$M_{i,j} = \delta_{i,j+1} - 2\delta_{i,j} + \delta_{i,j-1}. \tag{51}$$

The solution is

$$\langle\tau(t)\rangle = G(t,0)\langle\tau(0)\rangle \tag{52}$$

where we defined a generating function (matrix)

$$\frac{dG(t,t')}{dt} = MG(t,t') \quad \text{with} \quad G(t',t') = \mathbb{I} \quad \text{(identity)} \tag{53}$$

We denote the elements of $G(t,t')$ as $g_{i,j}(t,t')$.

From the rates one finds that the average of integrated current between sites $i$ and $i+1$ as

$$\frac{\langle Q_i(t)\rangle}{dt} = \langle\tau_i(t)\rangle - \langle\tau_{i+1}(t)\rangle. \tag{54}$$

The solution is

$$\begin{aligned}\langle Q_i(t)\rangle &= \int_0^t dt'\,[\langle\tau_i(t')\rangle - \langle\tau_{i+1}(t')\rangle] \\ &= \int_0^t dt' \sum_j [g_{i,j}(t',0) - g_{i+1,j}(t',0)]\,\langle\tau_j(0)\rangle\end{aligned} \tag{55}$$

*Auto-correlation:* Starting from the rates one can write

$$\frac{\langle Q_j(t)Q_i(t')\rangle_c}{dt} = \langle \tau_j(t)Q_i(t')\rangle_c - \langle \tau_{j+1}(t)Q_i(t')\rangle_c \tag{56}$$

For the right hand side we use

$$\langle \tau_j(t)Q_i(t')\rangle_c = \sum_k g_{j,k}(t,t')\langle \tau_k(t')Q_i(t')\rangle_c. \tag{57}$$

This leads to

$$\frac{d\langle Q_j(t)Q_i(t')\rangle_c}{dt} = \sum_k \left[g_{j,k}(t,t') - g_{j+1,k}(t,t')\right]\langle \tau_k(t')Q_i(t')\rangle_c \tag{58}$$

Integrating

$$\langle Q_j(t)Q_i(t')\rangle_c = \langle Q_j(t')Q_i(t')\rangle_c + \sum_k \int_{t'}^{t} dt''\left[g_{j,k}(t'',t') - g_{j+1,k}(t'',t')\right]\langle \tau_k(t')Q_i(t')\rangle_c \tag{59}$$

The two-time correlation of current is expressed in terms of two equal time correlations: the $\langle \tau_j(t)Q_i(t)\rangle_c$ and $\langle Q_j(t)Q_i(t)\rangle_c$. In the following we determine them.

Starting from the rates one finds

$$\frac{d\langle \tau_j(t)Q_i(t)\rangle_c}{dt} = \sum_k M_{j,k}\langle \tau_k(t)Q_i(t)\rangle_c + A_{j,i}(t) \tag{60}$$

with

$$A_{j,i}(t) = \begin{cases} \langle [\tau_{j+1}(t) - \langle \tau_j(t)\rangle]\,[\tau_i(t) - \tau_{i+1}(t)]\rangle, & \text{for } j = i \\ \langle [\tau_{j-1}(t) - \langle \tau_j(t)\rangle]\,[\tau_i(t) - \tau_{i+1}(t)]\rangle, & \text{for } j = i+1 \\ \langle \tau_j(t)\tau_i(t)\rangle_c - \langle \tau_j(t)\tau_{i+1}(t)\rangle_c, & \text{for others} \end{cases} \tag{61}$$

Integrating one obtains

$$\langle \tau_j(t)Q_i(t)\rangle_c = \sum_{k=1}^{L}\int_0^t dt' g_{j,k}(t,t')A_{k,i}(t') \tag{62}$$

This leads to for $T_2 > T_1$:

$$\begin{aligned}\langle Q_j(T_2)Q_i(T_1)\rangle_c &= \langle Q_j(T_1)Q_i(T_1)\rangle_c \\ &+ \sum_k\sum_\ell \int_{T_1}^{T_2} dt \int_0^{T_1} dt'\left[g_{j,k}(t,T_1) - g_{j+1,k}(t,T_1)\right] g_{k,\ell}(T_1,t')A_{\ell,i}(t').\end{aligned} \tag{63}$$

The last quantity to determine is $\langle Q_j(T)Q_i(T)\rangle_c$. We show for $j \neq i$

$$\frac{d\langle Q_j(T)Q_i(T)\rangle_c}{dT} = \langle \tau_j(T)Q_i(T)\rangle_c - \langle \tau_{j+1}(T)Q_i(T)\rangle_c + \langle \tau_i(T)Q_j(T)\rangle_c - \langle \tau_{i+1}(T)Q_j(T)\rangle_c \tag{64}$$

whereas for $j = i$ we get

$$\frac{d\langle Q_i^2(T)\rangle}{dT} = 2\left[\langle \tau_i(T)Q_i(T)\rangle_c - \langle \tau_{i+1}(T)Q_i(T)\rangle_c\right] + \langle (\tau_i(T) - \tau_{i+1}(T))^2\rangle. \tag{65}$$

Integrating over time and using (62) we get for $j \neq i$

$$\langle Q_j(T)Q_i(T)\rangle_c = \sum_{k=1}^{L}\int_0^T dt \int_0^t dt'\left[g_{j,k}(t,t') - g_{j+1,k}(t,t')\right] A_{k,i}(t')$$

$$+\sum_{k=1}^{L}\int_0^T dt\int_0^t dt'\,[g_{i,k}(t,t')-g_{i+1,k}(t,t')]\,A_{k,j}(t')+\delta_{j,i}\int_0^T\langle[\tau_i(t)-\tau_{i+1}(t)]^2\rangle \quad (66)$$

The (66) along with (63) is the solution for the two-time correlation in symmetric exclusion process. For an explicit solution one needs to use the result of the two-point correlations of the occupation variables given in [Derrida, Gerschenfeld, Bethe ansatz Paper]. For the special case of $\rho_a = 1$ and $\rho_b = 0$ the expression is simple

$$\langle\tau_i(t)\rangle = \int \frac{dz}{2\pi\hat{i}z} e^{t(z+1/z-2)} \frac{z^i}{1-z} \quad (67)$$

$$\langle\tau_i(t)\tau_j(t)\rangle_c = \int \frac{dzdz'}{4\pi^2 zz'} e^{t(z+1/z+z'+1/z'-4)} \frac{z^i z^j}{zz'+1-2z'} \quad (68)$$

This leads to an exact expression of the two-time correlation between any two sites. For the large time behaviour one may use the scaling function given in Eq(37) in [Derrida, Gerschenfeld].

*Simple case:* A simple example is the equilibrium case where $\rho_a = \rho_b = \rho$. Here the two-point correlation function is independent of time and

$$\langle\tau_j(t)\tau_i(t)\rangle_c = \rho(1-\rho)\delta_{i,j}. \quad (69)$$

Show that the time dependence is same as found using macroscopic approach in (49).

## Appendix A. A closed form solution for $P$ and $Q$ (I have to check carefully).

$$P(x,t) = \begin{cases} P_1(x,t) & \text{for } x<0 \\ P_2(x,t) & \text{for } x>0 \end{cases}, \quad (A.1)$$

where

$$P_1(x,t) = 1 + \int_t^T d\tau\, \frac{1}{2}\text{erfc}\left(\frac{-x}{\sqrt{4(\tau-t)}}\right)\, \lambda(\tau)\exp\left(\frac{1}{2}\int_\tau^T dr\ \lambda(r)\right)$$

$$P_2(x,t) = \exp\left(\int_t^T dr\ \lambda(r)\right)\left[P(x,T) - \int_t^T d\tau\, \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{4(\tau-t)}}\right)\, \lambda(\tau)\exp\left(-\frac{1}{2}\int_\tau^T dr\ \lambda(r)\right)\right].$$

Similarly, the solution for $Q(x,t)$ yields, The solution

$$Q(x,t) = \begin{cases} Q_1(x,t) & \text{for } x<0 \\ Q_2(x,t) & \text{for } x>0 \end{cases}, \quad (A.2)$$

where

$$Q_1(x,t) = P(x,0) + \int_0^t d\tau\, \frac{1}{2}\text{erfc}\left(\frac{-x}{\sqrt{4(t-\tau)}}\right)\, \lambda(\tau)\exp\left(\frac{1}{2}\int_0^\tau dr\ \lambda(r)\right)$$

$$Q_2(x,t) = \exp\left(\int_0^t dr\ \lambda(r)\right)\left[P(x,0) - \int_0^t d\tau\, \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{4(t-\tau)}}\right)\, \lambda(\tau)\exp\left(-\frac{1}{2}\int_0^\tau dr\ \lambda(r)\right)\right].$$

## References