3
Introduction

3.1 Self-organized criticality (SOC)

One of the most striking aspects of physics is the simplicity of its laws. Maxwell's equations, Schrodinger's equation, and Hamiltonian mechanics are simple and expressible in few lines. However every place we look, outside the textbook examples, we see a world of amazing complexity: huge mountain ranges, scale free coastlines, the delicate ridges on the surface of sand dunes, the interdependencies of financial markets, the diverse ecologies formed by living organisms are few examples. Each situation is highly organized and distinctive, but extremely complex. So why, if the basic laws are simple, is the world so complicated? The idea of Self Organized Criticality was born aiming to give an explanation for this ubiquitous complexity [J.98]. In this chapter the basic concepts related to SOC, that will be important for this thesis, are introduced.

The examples, cited above, share a common feature: a power-law tail of the correlations. Consider the two point correlation of a quantity $\Delta h(\mathbf{x}) = h(\mathbf{x}) - \bar{h}$, where $h(\mathbf{x})$ is the height at a place $\mathbf{x}$ in a mountain range, and $\bar{h}$ is its average value. The function $\langle \Delta h(\mathbf{x}+\mathbf{r}) \Delta h(\mathbf{x}) \rangle$ increases as r^{δ}, with the exponent δ varying very little for different mountain ranges. Similar distribution with extended tails is observed in many other natural phenomena: Gutenberg-Richter law in earth quake [GR56], Levy distribution in stock market price variations [Bak96a], Hacks law in River networks [DR99, BCF$^+$01] etc. Such power-law distributions entail scale invariance — there are no macroscopic spatial scales other than the system size, in terms of which one can describe the system, making it complex.

Such features are familiar to physicist in equilibrium systems undergoing phase transition. In standard critical phenomena there are control parameters such as temperatures, magnetic field, which requires to be fine tuned by an external agent, to reach the critical point. This is unlikely to happen in naturally occurring processes such as formation of mountain ranges, earth quakes or even stock markets. These are non-equilibrium systems brought to their present states, by their intrinsic dynamics — and not by a delicate selection of temperature, pressure or similar control parameters. [1].

In the summer of 1987, Bak, Tang and Wiesenfeld(BTW) published a paper [BTW87] proposing an explanation to such ubiquitous scale invariance. They argued that the dynamic which gives rise to the robust power-law correlations seen in the non-equilibrium steady states in nature must not involve any fine tuning of parameters. It must be such that the systems under their *natural evolution* are driven to a state at the boundary between the stable and unstable states. Such a state then shows long range spatio-temporal fluctuations similar to those in equilibrium critical phenomena. The complex features appear spontaneously due to a cooperative behavior between the components of the system. They called this self-organized criticality as the system self-organizes to its critical steady state.

SOC nicely compliments the idea of chaos. In the latter, dynamical systems with a few degrees of freedom, say as little as three, can display extremely complicated behavior. However, a statistical description of this randomness is predictable in the sense that, the signals have a white noise spectrum, and not a power law tail. A Chaotic system has little memory of the past, and it is easy to give a statistical description of such behavior. In short, chaos does not explain complexity. On the other hand, in SOC, generally, we start with systems of many degrees of freedom, and find a few general features which are also statistically predictable, but has a power-law spectra leading to complex behavior. In certain dynamical systems, *e.g.*, logistic maps, there are points (the Feigenbaum point [Fei78]) in the parameter space, which separates states with a predictable periodic behavior and chaos. At this transition point there is complex behavior, with power-law correlations. SOC gives description of how systems, under their own dynamics, without external monitoring, reaches this very special point ("edge of chaos"), explaining the robust complex behavior in natural systems.

In the book "How nature works?", Per Bak gives various kinds of natural examples of SOC, of which the canonical one is the sandpile. On slowly adding grains of sand to an empty table, a pile will grow until its slope becomes critical and avalanches start spilling over the sides. If the slope is too small, each grain just stays at the place where it lands or creates a small avalanche. One can understand the motion of each grain in terms the local properties, like place, the neighborhood around it etc. As the

[1] Per Bak, in his book [Bak96b], puts this in an interesting comment—"The nature is operated by a 'blind watchmaker' who is unable to make continuous fine adjustments"

process continues, the slope of the pile become steeper and steeper. If the slope becomes too large, a large catastrophic avalanche is likely, and the slope will reduce. Eventually, the slope reaches a critical value where there are avalanches of all sizes. At this point, the system is far out of balance, and its behavior can no longer be understood in terms of the behavior of localized events. The system is invariably driven towards its critical state.

3.2 Theoretical models

In order to have a mathematical formulation of SOC, BTW studied a so-called cellular automata known as the sandpile model [BTW87], which is discrete in space, time and in its dynamical variables. The model is defined on a two dimensional square lattice where each site i has a state variable z_i referred as height, which takes only positive integer values. This integer can be thought of as representing the amount of sand at that location or in another sense it represents the slope of the sandpile at that point. Neither of these analogies is fully accurate, the model has aspects of both. One should consider it as a mathematical model of SOC, rather than an accurate model of physical sand.

A set of local dynamical rules defines the evolution of the model: At each time step a site is picked randomly, and its height z_i is increased by unity. In this thesis, this step will be referred as the *driving*. If the height now is greater than or equal to a threshold value $z_c = 4$, the site is said to be unstable. It relaxes by toppling whereby four sand grains leave the site, and each of the four neighboring sites gets one grain. If there are any unstable sites remaining, they too are toppled, all in parallel. In case of toppling at a site at the boundary of the lattice, grains falling outside the lattice are removed from the system. This process continues until all sites are stable. This completes one time step. Then, another site is picked randomly, its height is increased by 1, and so on.

The following example illustrates the dynamics. Let the lattice size be 3×3 and suppose at some time step the following configuration is reached where all sites are stable.

2	3	2
3	3	0
1	2	3

We now add a grain of sand at randomly selected site: let us say the central site is chosen. Then the configuration becomes the following

2	3	2
3	4	0
1	2	3

The central site is not stable, and therefore it will topple and the configuration becomes

2	4	2
4	0	1
1	3	3

.

This configuration has two unstable sites, so both will topple in parallel. Since these are at the boundary, two grains will be lost, on toppling. The new result is

4	0	3
0	2	1
2	3	3

,

and further toppling leads to

0	1	3
1	2	1
2	3	3

This is a configuration with all sites stable. One speaks in this case of an avalanche of size $s = 4$, since there are four topplings. Another measure is the number of steps required for relaxation, which in this case is $t = 3$. For large lattices, in the steady state, the distribution of avalanche sizes and durations display a long power-law tail, with an eventual cutoff determined by the finite size of the system.

Since the original sandpile model by BTW a large number of variations of the model have been studied (see [Dha06, J.98] for reviews). These are mostly extended systems with many components, which under steady drive reaches a steady state where there are irregular burst like relaxations and long ranged spatio temporal correlations. It is to be noted that in these models the complexity is not contained in the evolution rules itself, but rather emerges as a result of the repeated local interactions among different variables in the extended system.

In the rest of this chapter, I will introduce three of the most studied models of sandpile and the techniques used to analyze them.

3.2.1 Deterministic abelian Sandpile Model (DASM)

This is the most studied model due to it analytical tractability. In a series of papers, Deepak Dhar and his collaborators have shown that this model has some remarkable mathematical properties. In particular, the critical state of the system has been well characterized in terms of an abelian group. In the following I will generally follow the discussion in [Dha06].

The model is a generalized BTW model on any general graph with N sites labeled by integers $1, 2, 3 \cdots N$. To make things precise, I will start with some definitions. A configuration C for the sandpile model is specified by a set of integer heights $\{z_i\}$ defined on the N sites of the graph. We denote a threshold value of the height at a site i as z_i^c. The system is driven like the BTW model by adding one sand grain at a randomly chosen site which increases the height at that site by 1. The toppling rules are specifies by a $N \times N$ toppling matrix Δ such that on toppling at site i, heights at all sites are updated according to the rule:

$$z_j \to z_j - \Delta_{i,j} \text{ for every j.} \tag{3.1}$$

For example in the BTW model on a square lattice

$$\Delta_{i,j} = \begin{cases} 4 & \text{for } i = j \\ -1 & \text{for } i,j \text{ nearest neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

Evidently the matrix Δ has to satisfy some conditions to ensure that the model is well behaved. These are

1. $\Delta_{i,i} > 0$, for all i. (Height decreases at the toppled site)
2. $\Delta_{i,j} \leq 0$, for all $j \neq i$. (Heights at other sites are increased or unchanged)
3. $\sum_j \Delta_{i,j} \geq 0$ for all i. (Sand is not generated in toppling)
4. Each site is connected through toppling events to at least one site where sand can be lost, such as the boundary.

Without loss of generality we choose $z_i^c = \Delta_{i,i}$ (This only amounts to defining the reference level for the height variables).

With this convention, if all z_i are initially non-negative they will remain so, and we restrict ourself to configurations C belonging to that space, denoted by Ω. Let $\mathbb{S}$ be the space of stable configurations denoted by C_s where the height variables at each site are below threshold. The property 4 above ensures that stability will always be achieved in a finite time.

We formalize the addition of sand to a stable configuration by defining an "addition operator" a_i so that $a_i C_s$ is the new stable configuration obtained by taking $z_i \to z_i + 1$ and then relaxing.

The mathematical treatment of ASM relies on one simple property it possesses: The order in which the operations of particle addition and site toppling are performed does not matter. Thus the operators a_i commute *i.e.*

$$a_i a_j C_s = a_j a_i C_s \quad \text{for every } i,j \text{ and } C_s. \tag{3.3}$$

The proof uses the linearity of the toppling processes [Dha06]. In the relaxation processes represented by the two sides of the above equation, the order of topplings can be changed, but the final configurations are equal. An example of this abelian nature is the process of long addition of multi-digit numbers. In this example the toppling process is like carrying.

Note that, there are some "garden of Eden" configurations that once exited can not be reached again. For example, in the BTW model on square lattice, system can never reach a state with two adjacent $z_i = 0$. This is because in trying to topple a site to zero, the neighbor gains a grain, and vice versa. This leads to the definition of the recurrent state space $\mathcal{R}$ which consists of any stable configuration that can be achieved by adding sand to some other recurrent configuration. This set is not empty since one can always reach a minimally stable configuration defined by having all $z_i = z_c - 1$.

Dhar proved [Dha90] another remarkable property that the addition operators a_i have unique inverses when restricted to the recurrent space; that is, there exists a unique operator a_i^{-1} such that $a_i \left(a_i^{-1} C_s \right) = C_s$ for all C_s in $\mathcal{R}$. This can be easily seen from the fact that there are finite number of configurations in $\mathcal{R}$, so for some positive period p, $a_i^p C_s = C_s$ with C_s a recurrent configuration. Using the abelian property it can be shown that the period p is same for all $C_s \in \mathcal{R}$. Then a_i^{p-1} is the inverse for a_i.

These properties of a_i have some interesting consequences [Dha90]. One is that in the steady state all the recurrent configurations are equally probable. Also, the number of recurrent states is simply the determinant of the toppling matrix Δ. For large square lattices of N sites this determinant can be found easily by Fourier transform. In particular, whereas there are 4^N stable states, there are only $(3.2102\cdots)^N$ recurrent states. Thus starting from an arbitrary state and slowly adding sand, the system self-organizes to an exponentially small subset of states, which are the attractor of this dynamics.

There are many more interesting properties of the DASM, *e.g.*, using a burning algorithm [MD92], it is possible to test whether an arbitrary configuration is recurrent. Using this algorithm it can also be shown that the model is related to statistics of spanning-trees on the lattice, as well as with the $q \to 0$ limit of the Potts model [MD92, Dha06]. As several results are known about spanning tree these equivalence help in relating properties of DASM to known properties of spanning trees.

In spite of these interesting mathematical properties, the exponents characterizing the power-law tail in the distribution of avalanches are still difficult to determine analytically on most lattices, and computer simulations are still needed. In fact, on a square lattice, the numerical values estimated by different people have shown a wide range of values. It has been argued that the simple finite size scaling does not work for the avalanche distribution and instead it has a multi-fractal character [KNWZ89]. In some simpler quasi-one dimensional lattices it has been shown that simple linear combination of two scaling forms provides an adequate description [AD95].

For higher dimensional lattices it has been shown by Priezzhev that the upper critical dimension for the models is 4 [Pri00]. For lattices of dimension above 4, the avalanche exponents take mean field values and can be deduced from the exact solution of the model on a Bethe lattice [DM90].

3.2.2 Zhang model

The Zhang model, introduced by Zhang in 1989, differs from the DASM in two aspects: first, the height variables z_i are continuous and takes non-negative real values. A site is unstable if its height is above threshold, and it topples by equally dividing its entire content among its nearest neighbors, and itself becoming empty. Second, the external perturbation is not by adding height 1, but by a quantum chosen randomly from an interval $[a, b)$, where $0 \leq a < b$ are positive real numbers.

Here, is an example of the Zhang model in one dimension. Let the threshold height is $z_i^c = 1.5$, same for all i, and an initial configuration is

0.8	1.4	0

Now a time step begins by an addition to a random site, of a random amount chosen from the interval $[0, 1.5)$. Let the amount is 1.0 and the site is the central site. After addition the result is

0.8	2.4	0

Because the middle site is unstable, an avalanche starts:

2.0	0	1.2	$\rightarrow$	0	1.0	1.2

In case of two or more unstable sites, all are toppled in parallel.

Since the addition amount is random, a stable site could in principle have any height between zero and the threshold and the stationary distribution could be very different from that of the DASM, where only discrete values are encountered. Nevertheless, when one simulates the model on large lattices in one and two dimensions, the stationary heights tend to concentrate around nonrandom discrete values. This is known as the "emergence of quasi-units" [Zha89]. It appears that altering the *local* toppling rules of the DASM, does not have that much effect on the *global* behavior after all, if the lattice size is large.

This behavior led to the conjecture that in the thermodynamic limit the critical behavior is identical to that of DASM. However, due to the changed toppling rules, the dynamics is no longer abelian, and determining the steady state is quite difficult, even in one-dimension. In fact, Blanchard *et. al.* have shown that the probability distribution of heights in the steady state, even for the two site problem, has a multi-fractal character [BCK97].

This status was unchanged for over a decade until Fey *et. al.* showed that on a one-dimensional lattice, for some specific choice of the amount of addition, the toppling becomes abelian. Using this they showed that, indeed, the model is on the same universality class of the DASM. However, the analysis in two dimension is still an open problem.

3.2.3 Manna model

Another important class of the sandpile models are with stochastic toppling rules. The first model of its kind was studied by Manna in 1991, and is known as the Manna model [Man91].

The evolution rules of this sandpile in d-dimensions are very similar to the ones defined for the DASM. In fact, the driving rule and the dissipation rules at the "boundary" remain the same. But in a toppling, an unstable site redistributes *all the sand grains* between sites randomly chosen amongst its $2d$ nearest neighbors.

$$\begin{aligned} z_i &\to 0 \\ z_j &\to z_j + 1 \quad \text{for } z_i \text{ sites chosen randomly amongst n.n. of } i. \end{aligned}$$

The randomness in the evolution rule is a relevant change in the dynamics, which makes it non-abelian. It is possible to get back the abelianness by a simple modification in the toppling rule, which I will discuss in detail in the later chapters. However, the addition operators defined appropriately do not form a group anymore and this makes the analysis less tractable even for a linear chain.

The steady state is very different from its deterministic counter part *e.g.* on a simple linear chain the different recurrent states are not equally probable, unlike the deterministic model. Also the avalanche distribution can be satisfactorily described by simple finite size scaling. Another evidence of the differences between these models is in the way the avalanches spread over the lattice [MBS98b]. The distribution of number of toppling per site in a typical avalanche for both DASM and Manna model on a square lattice are shown in Fig. 3.2.3. For the DASM, one can see a shell structure in which all sites that toppled T times form a connected cluster

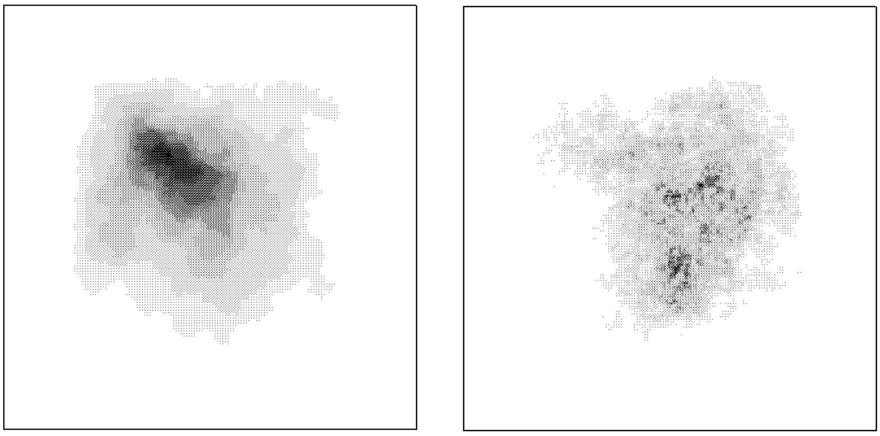

Figure 3.1: Number of toppling per site for a typical avalanche in (a) DASM and (b) Manna model. The darker shades denote more topplings. (Courtesy [MBS98b])

with no holes, and these sites are contained in the cluster of sites that toppled $T-1$ times, and so on. On the other hand, the toppling distribution exhibits a random avalanche structure with many peaks and holes.

For many years, the universality of the manna model was a controversial question. At present there are convincing numerical evidences that in dimension up to 3, they have a different critical behavior, from its deterministic counterpart, with a different set of critical exponents. However in dimensions $d \geq 4$, both DASM and Manna model take the same mean-field values of critical exponents.

3.3 Universality in the sandpile models.

Since the work by BTW, a large number of different models have been studied *e.g.* sandpile models with many variations of the BTW toppling matrix [KNWZ89] or sand grain distribution rules [MZ96], stochastic topplings [Man91], with activity inhibition [MG97], continuous height models [Zha89], loop erased random walk [DD97], Takayasu aggregation model [Tak89], train model [dSV92, PB96], non-abelian sandpile directed sandpile model [LLT91, PZL+05, Ali95a, GH02], forest-fire model [DS92], Olami-Feder-Christensen model [OFC92] *etc* (and many more could have been defined). Most of these models could only be studied numerically, and for a while it seemed that each new variation studied belong to a new universality class each with its own set of critical exponents. It is a fair question

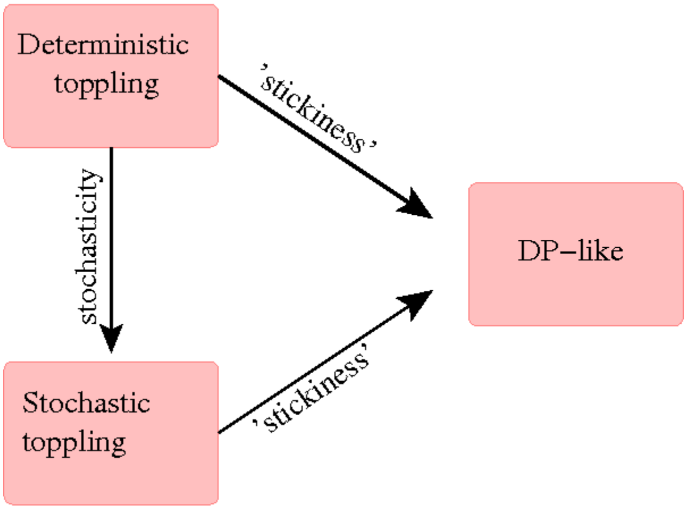

Figure 3.2: A schematic flow diagram of renormalization group flows between different fixed points of sandpile models.

to ask, what is the generic behavior?

Although this question is not yet resolved completely, by now, there has been a fair amount of understanding of this problem. The universality classes with renormalization group flow in these models can be summarized in the Fig. 3.2.

There are sufficient numerical evidence that sandpile models with deterministic toppling rules (DASM) and stochastic toppling rules (Manna model) constitute different universality classes. There are also several other model which show critical exponents different from these two [Sne95, BS93, GZ96]. They are related to the directed percolation (DP) [Kin85], which describes the active-absorbing state phase transition in a wide class of reaction-diffusion systems. The activity in avalanches in sandpile can grow, diffuse or die, and any stable configuration is an absorbing state. Thus one would expect that in general the sandpile should belong to the universality class of active-absorbing state transition with many absorbing states [RMAS00]. However, these models do not involve any conserved fields. In Manna and DASM-type models, it is this presence of conservation laws of sand that makes the critical behavior different from DP [VDMnZ98].

Recently, the effect of non-conservation has been explicitly studied [MD02, MD07] by introducing stickiness in the toppling rules (*i.e.* there is small probability that the incoming particles to a site get stuck there, and do not cause any toppling until the next avalanche hits the site, thus in effect there is no conservation of grains within an avalanche). It has been argued that as long as the sand grains have non-zero stickiness, the distribution of avalanche sizes follows directed percolation exponents. The DASM, and the stochastic Manna-type models are unstable to this pertur-

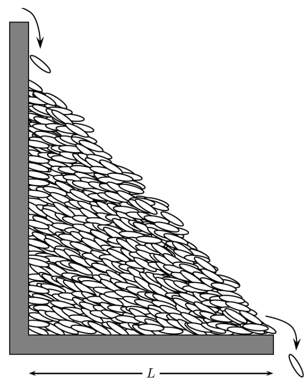

Figure 3.3: A schematic of a rice-pile. The elongated shapes of the rice grains reduces the inertial effect in an avalanche. (Courtesy K. Christensen.)

bation, and the renormalization group flows are directed away from them to the directed percolation fixed point, as schematically shown in the figure. This picture is exactly verified in *directed* sandpiles. However, the argument is less convincing for undirected models, and the issues is not settled [BRC⁺06].

3.4 Experimental models of SOC

Soon after the sandpile model was introduced, several experimental groups measured the size distribution of avalanches in granular materials. Unfortunately, real sandpile do not seem to behave as the the theoretical models. Experiments show large periodic avalanches separated by quiescent states with only limited activity. While for small piles one could try to fit the avalanche distribution with power-law over a limited range, the behavior would eventually cross over, on increasing the system size, to a state which is not scale-invariant [JLN89, JNB96]. It is later realized that inertia of rolling grains is the reason for non-criticality. A large avalanche propagating over a surface with slope θ_c scours the surface, and takes away materials from it. The final angle, after the avalanche has stopped, is bellow θ_c. So if we want to see power-law avalanches, we have to minimize the effect of inertia of the grains. This is achieved in an experiment in Oslo by using rice grains. Because of the elongated shape of the rice grains (Fig. 3.3) frictional forces are stronger and these poured at very small rate gave rise to a convincing power-law avalanche distribution [FCMS⁺96].

A similar power law distribution of avalanche sizes are obtained in motion of domain walls in ferro-magnets [DBM95, ZCDS98] and flux lines in type II superconductors [FWNL95, ORN98]. A more recent experimental realization of SOC is obtained in suspensions of sedimenting non-brownian particles by slow periodic shear [CGMP09].

It is worth mentioning that SOC has been invoked in several other situations in geophysics (atmospheric precipitation [PN06], river pattern due to erosion [TI92], landslides [MT99]), biology (neural-network [LHG07]), economics (stock-market crashes [SS99]) and many more. I have deliberately chosen the above experimental examples for which experimental observations are accurate and reproducible.

3.5 Remarks

Originally, SOC was proposed with the aim of providing an explanation of the ubiquitous complexity in nature [BTW87]. The abundance of fractal structures in nature, temporal as well as spatial, was considered to be an effect of a generic tendency — pertinent to most many-body systems — to develop by themselves in to a critical scale-invariant state.

However, certainly not all systems that organize themselves into one specific state will, when gently driven, exhibit scale invariance in that self-organized state. The experiments of real sandpiles referred earlier are a prime example. Neither is all observed power law behavior are an effect of dynamical self-organization into a critical state. The work by Sethna and co-workers on Barkhausen noise [SDK+93] is an interesting example of this, what Didier Sornette has called “Power laws by sweeping of an instability” [Sor94].

Since the introduction of the idea, a large amount of discussion went into understanding the minimum conditions for a model to be self-organized critical. Though a broad picture has emerged in last two decades, it is still not complete and controversial. In the rest of this section, I will discuss two of the most discussed properties, using both examples and counter examples.

- **Slow driving limit.** There is a strong belief in the community that an essential ingredient of SOC is *slow driving* (driving and dynamics operating at two infinitely separated timescales, *i.e.* avalanches are instantaneous relative to the time scale of driving). This idea got widely accepted after an argument given by Dickman *et. al.*.[RMAS00] They argued that the dynamics in the sandpile model implicitly involve tuning of the *density* of grains to a value where a phase transition takes place between an active state, where topplings take place, and a quiescent “absorbing” state. When the system is quiescent, addition of new particles increases the density. When the system is active, particles are lost to the sinks via toppling, decreasing the density. The slow driving ensures that these two density changing mechanisms balance one another out, driving the system to the threshold of instability.

However in the Takayasu model of aggregation [Tak89] the driving is fast. A simple example, it can be defined on a linear chain on which particles are continuously injected, diffuse and coalesce. One can write the explicit rules as follows:

- At each time step, each particle in the system moves by a single step, to the left or to the right, taken with equal probability, independent of the choice at other sites.
- A single particle is added at every site at each time step.
- If there are more than one particle at one site, they coalesce and become a single particle whose mass is the sum of the masses of the coalescing parts. In all subsequent events, the composite particles acts as a single particle.

The probability distribution of total mass at a randomly chosen site, has power law tail, with an upper cutoff that increases with time. This is a signature of criticality. The analogue of avalanches in this model is the event of coming together of large masses. In fact, it can be shown [Dha06] that the model is equivalent to a directed version of sandpile. In this example, it is clear that the driving is fast, and the rate is comparable to the local movements of the particles.

• **Conservation.** Conservation of grains is also considered as a key property for the criticality to emerge in sandpile models. A simple intuitive argument goes as follows: the sand grains introduced in the pile can dissipate only by reaching the diffusive "boundary" of the lattice. Owing to this and because of the vanishing rate of sand addition, arbitrarily large avalanches (of all possible sizes) must exist for an arbitrarily large system size, yielding a power-law size distribution. In contrast, in the presence of non-vanishing bulk dissipation, grains disappear at some finite rate, and avalanches stop after some characteristic size determined by the dissipation rate. This clearly says that bulk dissipation is a relevant perturbation in the *sandpile dynamics* and breaks the criticality [BMn08].

  There are also some other models of SOC like Forest fire [DS92], OFC model [OFC92] where it was shown, mostly numerically, that non-conservation in the dynamics leads to non-critical steady state.

  However, extrapolating these results and considering conservation as a neccesarry criteria for SOC, in general, is not correct. A model which is clearly non-conservative and still, when slowly driven displays power-law in the avalanche size distribution is discussed in [Sad10]. Another two non-conservative models of SOC are a sandpile model with threshold dissipation[Ali95b], and Bak-Sneppen model of evolution [BS93].

Finally, one could ask: Has SOC, taught us anything about the world that we did not know prior to it? Jensen addresses this question very nicely in his book. The most important lesson is that, in a great variety of systems, particularly for slowly-driven-interaction- dominated-threshold systems, it is misleading to neglect fluctuations. In these systems, sometimes, the fluctuations are so large that the fate of a major part of the system can be determined by a single burst of activity. Dinosaurs may have become extinct simply as a result of an intrinsic fluctuation in a system consisting of a highly interconnected and interacting web of species; there may be no need for an explanation in terms of external bombardment by meteorites. *Fluctuations are so large that the "atypical" events decides the future of the system*. This new insight is sufficiently important to justify and inspire more theoretical, and experimental research in SOC.

3.6 Overview of the later chapters

The work in this thesis ranges from characterizing the spatial patterns in sandpile model, to quasi-units in the stationary distribution of Zhang's model, and determining exact steady state of Manna model. The first three chapters in the following are about sandpile as a growth model, where we show how well-structured non-trivial patterns emerge at large length scales, due to local interactions in cellular lengths where the patterns are not obvious. In chapter 7 we discuss another emergent behavior in the Zhang model. Chapter 8 contains an operator algebraic analysis of the stochastic sandpile models. Here is a brief summary of these chapters.

Chapter 4: While a considerable amount of research went into characterizing the universality classes of sandpile models and understanding the mechanism of SOC, very limited work is done about spatial patterns in sandpile models. Such patterns were noted around the time when sandpile was first introduced [LKG90]. Yet, very little is known about them.

This chapter is devoted to the study of a class of such spatial patterns produced by adding sand at a *single* site on an *infinite* lattice with initial periodic distribution of grains and then relaxing using the DASM toppling rules. We present a complete quantitative characterization of *one* such patterns. We show that the spatial distances in the asymptotic (in the limit when large number of grains are added) patterns produced by adding a large number of grains, can be expressed in terms solution of discrete Laplace's equation (discrete holomorphic functions [Duf56, Mer01, Lov04]) on a grid on two-sheeted Riemann surface.

We also discuss the importance of these patterns as a paradigmatic model of growth where different parts of the structure grow in proportion to each other, keeping their shape the same. We call these kind of growth as *proportionate growth*. We discuss the importance of such growth in real world examples.

Chapter 5: In this chapter we describe how the pattern changes in presence of absorbing sites, reaching which the grains get lost and no longer participate in the avalanches. We show that, again, the *asymptotic* pattern can be characterized in terms of discrete holomorphic functions, but on a different lattice. Similar effects of multiple sites of addition on the pattern are also calculated.

The most interesting effect of the sink sites is the change in the rate of growth of the pattern. In absence of sink sites the diameter of the pattern, suitably defined, increases as $\sqrt{N}$ where N is the number of sand grains added in the lattice. When the pattern grows with the sink sites inside, the growth rate of the diameter changes, in general, to N^{α}, where the exponent α depends on the sink geometries. For example, $\alpha = 1/3$, when the sink sites are along an infinite line adjacent to the site where grains are dropped. When the site of addition is inside a wedge of angle $\pi/2$ with the sink sites along the wedge boundary, this value of the exponent is $1/4$. We use an scaling argument and determine α, for some simple sink-geometries.

Chapter 6: The growth rate also depends on the arrangement of heights in the background, and this dependence is quite intriguing. When the initial heights are low *enough* at all sites, one gets patterns with $\alpha = 1/d$, in d-dimension. If sites with maximum stable height $(z^c - 1)$ in the starting configuration form an infinite cluster, we get avalanches that do not stop, and the model is not well-defined. In this chapter, we study backgrounds in two dimensions. We describe our unexpected finding of an interesting class of backgrounds, that show an intermediate behavior: For any N, the avalanches are finite, but the diameter of the pattern increases as N^{α}, for large N, with $1/2 < \alpha \leq 1$, the exact value of α depending on the background. It still shows proportionate growth. We characterize the asymptotic pattern exactly for one illustrative example.

Chapter 7: As mentioned, the Zhang model on one and two dimensional lattices displays a remarkable property: emergence of quasi-units, where the continuous heights, in spite of the randomness in the driving, are peaked around a few discrete non-random values. Fey *et. al* have shown that on a linear chain the width of the distribution vanishes in the infinite volume limit. However they did not show how it approaches zero.

In this chapter, we show that, the sequence of toppling of the continuous height variables, when suitably discretized, have an one-to-one relation with that of integer heights in the corresponding DASM. We use this relation to show that the width of the distribution of heights decreases in inverse power of the length of the chain. We also determine how the variance of height at a site, changes with position of the site along the length of the system.

Chapter 8: This chapter contains an algebraic approach of determining the steady state of a class of sandpile models with stochastic toppling rules. The original Manna model, as discussed in section 1.2.3, does not have the abelian property of its deterministic counterpart. However, a simple modification of the toppling rules makes the model abelian [Dha99c]. A similar construction is possible for other stochastic toppling rules. However, analysis of these models are still difficult as the corresponding addition operators (see section 3.2.1), in general, does not have an inverse, and are not diagonalizable.

We show that, in principle, the operators can be reduced to a Jordan block form, using the algebra satisfied by these. These are then used to determine the steady state of the models. We illustrate this procedure by explicitly determining the numerically exact steady for a stochastic model on a linear chain. Using the desktop computers at our disposal, we have been able to perform the calculation for systems of size ≤ 12 and also studied the density profile in the steady state.