Assignment 1

Problem 1: Central limit theorem for correlated gaussian variables.

In lectures we showed that for independent identically distributed random variables x_i their sum has a gaussian distribution for large n, when $p(x)$ follows certain conditions. This is the central limit theorem.

For correlated random variables such limit laws are difficult. One exception when x's are gaussian distributed.

Consider $\{x_1, x_2, \cdots, x_n\}$ are gaussian variables with arbitrary correlation $\langle x_i x_j \rangle$ and $\langle x_i \rangle = 0$. Their joint probability

$$p(x) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det C}} \, e^{-\frac{1}{2} x^t \cdot C^{-1} \cdot x}$$

where $x^t = (x_1 \; x_2 \; \cdots \; x_n)$ a vector

$[C]_{n \times n}$ covariance matrix with $C_{ij} = \langle x_i x_j \rangle$

Show that $M = \sum_{i=1}^{n} x_i$ has probability distribution

$$\mathcal{P}(M) = \frac{1}{\sqrt{2\pi \langle M^2 \rangle}} \, e^{-\frac{M^2}{2\langle M^2 \rangle}}$$

Problem 2: ~~[illegible]~~ log-normal distribution and Benford's law.

(a) Consider iid random variables $x \geq 1$ with a distribution $p(x)$ with finite mean and variance. Let $M_n = \prod_{i=1}^{n} x_i$

Using central limit theorem show that $P(M_n)$ for large n is asymptotically given by a log-normal distribution.

[Hint: a log-normal distribution

$$p(m) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{1}{m} \cdot e^{-\frac{\left(\log \frac{m}{m_0}\right)^2}{2\sigma^2}}$$

Find expression of σ^2 and m_0 in terms of $\langle x \rangle, \langle x^2 \rangle_c$.

(b) We see that $M_n \geq 1$. Let z_n be the first digit of M_n in decimal representation. For example $M_n = 325.87$ has $z_n = 3$. Clearly z_n can be integers from 1 to 9.

Show that for large n, ~~[illegible]~~

$$\text{Prob}(z_n) \simeq \log_{10}\left(1 + \frac{1}{z_n}\right).$$

(c) This limiting distribution is related to Benford's-law which predict's that the same probability distribution $p(z)$ ~~[illegible]~~ describes the first digit of any large data set, such as stock price. Can you justify this based on what you just derived?

[Hint: the problem is discussed in the book of Kardar, 1st vol. Marks will be based on how clearly you explain the steps]

(d) A modified log-normal distribution

$$f_a(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot e^{-\frac{(\log x)^2}{2}} \left\{ 1 + a \sin(2\pi \log x) \right\}$$

with $-1 \leq a \leq 1$.

What is the k-th moment $\langle x^k \rangle$?

[You can do the integration in Mathematica]

Show that $\langle x^k \rangle$ is independent of parameter a. This illustrates the fact that although all moments are finite, knowing them does not give the probability distribution function. Explain why is this. [Hint: generating function]

Problem 3: Cauchy distribution. In lectures we discussed that Cauchy distribution is a stable distribution for sums of random variables. Show this explicitely by following ~~the~~ steps below.

(a) Cauchy distribution

$$p(x) = \frac{1}{\pi} \frac{c}{(x-x_0)^2 + c^2} \quad \text{with } c > 0.$$

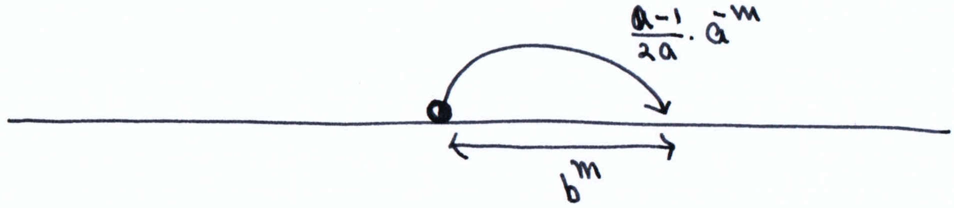
Use contour integration to show that $p(x)$ is normalized and that the characteristic function

$$g(k) = \langle e^{ikx} \rangle = e^{ikx_0 - c|k|}$$

(b) Use this to show that $s_n = \sum_{i=1}^{n} x_i$ has distribution

$$P(s_n) = \frac{1}{\pi} \cdot \frac{nc}{(s_n - nx_0)^2 + (nc)^2}$$

Problem 4: The Weierstrass walk. Consider a 1-d random walker (discrete time) which jumps on either side by step lengths $1, b, b^2, b^3, \ldots$ with probability $\frac{a-1}{2a} \cdot a^{-m}$ with $m = 0, 1, 2, 3, \ldots$.



This means probability of jump x is

$$p(x) = \frac{a-1}{2a} \sum_{m \geq 0}^{\infty} a^{-m} \left\{ \delta(x - b^m) + \delta(x + b^m) \right\}$$

with $a, b > 1$.

Position of the walker after n-steps is $S_n = \sum_{i=1}^{n} x_i$.

(a) Show that the characteristic function of $p(x)$ is

$$g(k) = \frac{a-1}{a} \sum_{m \geq 0} a^{-m} \cos(b^m k)$$

This is the famous Weierstrass function which is continuous everywhere but nowhere differentiable, when $b > a$.

Plot this function $g(k)$ for sum up to 10, for ($b = 2$ and $a = 1.5$) and ($b = 2$, and $a = 3.5$).

(b) Use the property that

$$g(k) = \frac{a-1}{a} \cos k + \frac{1}{a} g(bk)$$

to find the leading dependence for $k \to 0$. Show that

$$g(k) \sim \begin{cases} 1 - \frac{k^2}{2} \simeq e^{-\frac{1}{2}k^2} & \text{for } a > b^2 \\ 1 - \frac{1}{2}|k|^{\nu} \simeq e^{-\frac{1}{2}|k|^{\nu}} & \text{for } a < b^2 \end{cases}$$

with $\nu = \frac{\ln a}{\ln b}$.

(c) ~~[illegible]~~ Using the above result show that, for $a > b^2$, central limit theorem work, meaning M_n is distributed according to gaussian. For $a \leq b^2$, M_n has a Lévy distribution

$$P(M_n) \sim \frac{1}{M_n^{1+\alpha}} \quad \text{with} \quad \alpha = \frac{\ln a}{\ln b}.$$

This means, for $a > b^2$, there is normal diffusion, whereas for $a \leq b^2$ it is anomalous diffusion.

[Ref: ① Hughes et al, PNAS, 78 (1981), 3287.
② Chakravarti, Resonance, january, 50 (2004).]

Problem 5: Random matrices. Let $M \equiv (M_{ij})_{n\times n}$ real symmetric matrix with elements chosen randomly from distribution

$$P(M_{ij}) = \frac{1}{\sqrt{\pi}} e^{-M_{ij}^2} \quad \text{for } i < j$$

$$P(M_{ii}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{M_{ii}^2}{2}} \quad \text{for } i = j.$$

(a) Show that joint probability of all elements can be written as

$$P[\{M_{ij}\}] \equiv P(M) = A\, e^{-\frac{1}{2}\mathrm{tr}(M^2)}$$

A ← normalization constant.

This is called gaussian orthogonal ensemble (GOE) of random matrices.

(b) What is the distribution of $\mathrm{tr}\, M$ for large n?

[Hint: use CLT] What is $\langle (\mathrm{Tr}\, M)^2 \rangle$ and $\langle \mathrm{Tr}\, M \rangle$.

(c) There are n eigenvalues for M. It is shown by Wigner that probability density for an eigen value

$$P(\lambda) \equiv \frac{1}{n} \left\langle \sum_{i=1}^{n} \delta(\lambda - \lambda_i) \right\rangle$$

$$= \sqrt{\frac{2}{n\pi^2}} \left(1 - \frac{\lambda^2}{2n}\right)^{\frac{1}{2}} \quad \text{for large } n.$$

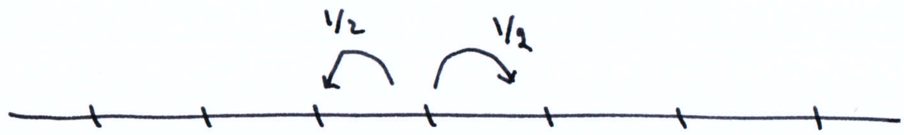
Use this to calculate $\left\langle \sum_i \lambda_i^2 \right\rangle$. [This is known as Wigner semi-circle law]

(d) Using results from (b)-(c), show that

$$\sum_{i<j} \langle \lambda_i \lambda_j \rangle \neq 0.$$

[This means eigenvalues of random matrix are correlated. Infact, it is known that there is Coulomb repulsion between eigenvalues. For more, see Majumdar, cond-mat/0701193]

Problem 6: Random walk in a disordered potential.



Consider a particle moving on a 1-d lattice, with continuous time. Each site has a random amount of potential E which is drawn from an exponential distribution

$$P(E) = \frac{1}{E_0} e^{-E/E_0} \qquad \text{with } E \geq 0.$$

When the particle is at site i with potential E_i, it waits for a time

$$t_i = \tau_0 \, e^{\beta E_i}$$

before making a jump to nearest neighbor. Here $\beta = \frac{1}{K_B T}$ and τ_0 is a microscopic time scale. This exponential waiting time comes from Arrhenius law.

(a) Find the waiting time distribution $p(t)$.

[Hint: use the relation between $p(t)$ and $p(E)$]

(b) Show that particle motion becomes sub-diffusive below a critical temperature T_c. What is this T_c?

[Hint: follow page 15 of second lecture note]

[<u>Remarks</u> (for you to think, not to answer.) What would happen if $p(E)$ is gaussian with $\langle E \rangle = 0$. Is there any ~~relatio~~ equivalence to Anderson localization?]