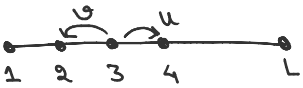
Exam

Wednesday, June 23, 2021 8:11 PM

Final exam for Adv statmech 2021.

Q1: A continuous time random walk on a 1d lattice with periodic boundary condition.



There are total L sites.

The jump rates $i \to i+1$ with u
$i \to i-1$ with v

(a) Write the continuous time Master equation. Write the explicit form of the Markov matrix W for $L=5$ sites.

(b) What is the largest eigenvalue and corresponding left eigenvector? Using symmetry argument, construct the corresponding right eigenvector including the correct normalization. Is detailed balance satisfied?

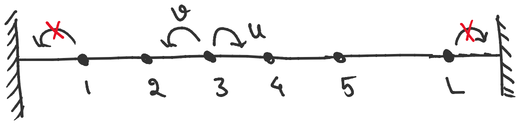
(c) Use an ansatz $|r\rangle = \frac{1}{L}\begin{pmatrix} e^{iq} \\ e^{i2q} \\ \vdots \\ e^{iLq} \end{pmatrix}$ for the right eigenvector. What are the allowed values of q for fixed L. What are the eigenvalues?

(d) What are the left eigenvectors with correct normalization?

(e) Let $P_K(t)$ is the probability for the random walker to be at site K at time t. For an initial condition $P_K(t=0) = \delta_{K,L}$, write an explicit solution for $P_K(t)$ as a summation over eigenvectors.

(f) In the $L \to \infty$ limit, write the sum as integral and then for large t complete the integration by considering leading contribution.
[Hint: asymptotic probability is Gaussian]

Q2: Consider another Continuous time random walk on L-sites with reflecting walls at $K=0$ and at $K=L+1$

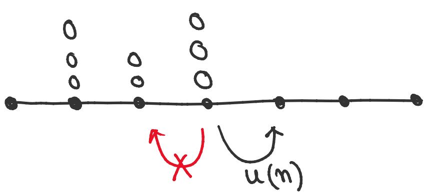


(a) What is the corresponding Markov matrix?

(b) Is detailed balance followed? What is the stationary probability distribution?

(c) Following earlier ansatz in Q1c, find the eigenvalues and eigenvectors.

Q3: Consider another variant of the problem in Q1.



The 1d lattice is of L sites with periodic boundary condition

There could be arbitrary number of particles at a site. If at a time a site has n particles, then a particle can jump from that site to only its right neighbor with rate $u(n)$.

Using pair wise balance show that the stationary state probability of a configuration with $\{n_1, n_2, \ldots, n_L\}$ particles is given by

$$P[n_1, \ldots, n_L] = \frac{1}{Z} f(n_1) f(n_2) \cdots f(n_L)\, \delta\left(N - \sum_{k=1}^{L} n_k\right)$$

where Z is normalization, N is total number of particles, and

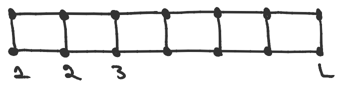
$$f(n) = \begin{cases} \dfrac{1}{u(1)\, u(2) \cdots u(n)} & \text{for } n \geq 1 \\ 1 & \text{for } n = 0 \end{cases}$$

Q4. Consider Ising spins on an $L \times L$ periodic square lattice with Hamiltonian

$$H = -J \sum_{ij} \sigma_{i,j}\, \sigma_{i+1,j}\, \sigma_{i,j+1}\, \sigma_{i+1,j+1} \qquad \text{with } \sigma_{ij} = \pm 1$$

Find the exact expression for the partition function.

Q5. Consider an Ising model on a $2 \times L$ lattice with nearest neigbor interactions



$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \qquad \text{with } \sigma_i = \pm 1$$

Periodic boundary condition only along the x direction.

(a) Construct the transfer matrix

(b) Find the corresponding eigenvalues.

(c) Find the free energy density in the thermodynamic limit.

(d) Find average magnetization.

Q6. Consider a Langevin equation on a v-shaped potential.

$$\dot{x} = -U'(x) + \eta(t) \qquad \text{with } \langle \eta(t) \rangle = 0, \ \langle \eta(t)\eta(t') \rangle = 2 k_B T\, \delta(t-t')$$

and $\qquad U(x) = |x|$

(a) Write the Fokker-Planck equation for this problem.

(b) By mapping to a schrödinger equation find eigenvalues and eigenfunctions (left and right) of the FP operator.

(c) For an initial condition $P_0(x) = \delta(x)$, write the solution for $P_t(x)$ in terms of eigenfunctions.

(d) What is the stationary state and what is the time scale to reach the stationary state?

Q7. Consider a coupled Langevin equation of two variables

$$\dot{x} = \alpha v + F(x) + \eta(t)$$

$$\dot{v} = -\lambda v + \xi(t)$$

with Gaussian white noises $\langle \eta(t)\rangle = 0$; $\langle \eta(t)\eta(t')\rangle = 2 k_B T\, \delta(t-t')$

$\langle \xi(t)\rangle = 0$; $\langle \xi(t)\xi(t')\rangle = 2D\, \delta(t-t')$

Following the Ito discretization, derive a path integral representation of the probability $P_t(x,v)$. Give the formula for action $S[x,v]$.

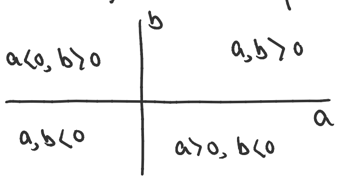
Q8. Tricritical point in Landau theory.

Consider a Landau free energy function

$$\mathcal{L}(\phi) = \frac{1}{2} a\phi^2 + \frac{1}{4} b\phi^4 + \frac{1}{6} c\phi^6$$

where $c > 0$ to ensure that $\mathcal{L}(\phi)$ grows for large $|\phi|$.

(a) Plot schematic of $\mathcal{L}(\phi)$ on four quadrants on (a,b) plane



You are free to use any plotting program, e.g. Mathematica. Make a qualitatively correct sketch. Be especially careful in the quadrant $a>0, b<0$.

(b) Determine the average value of the order parameter ϕ in all these quadrants.

(c) Estimating how the order parameter changes, draw a phase diagram on the (a,b) plane indicating nature of transitions.

(d) The $(a,b) \equiv (0,0)$ is a tricritical point. Near this point, by writing
$a = a_1 |T-T^*|$, $b = b_1 |T-T^*|$ where T^* is the tricritical temperature
show that average $\phi \sim |T-T^*|^{1/4}$.