# P-4811 Autumn 2015-2016
# Physics of Standard Model: Part I

**Homework # 2** **Due Date: 23/09/2015**

**2.1** Solve for $\mathcal{B}$ from the following equations

$$\mathcal{B}\eta = \cosh\theta\ \eta \tag{8}$$
$$\mathcal{B}^2 = \mathbb{I} + \sinh^2\theta\ \eta\eta^T\,, \tag{9}$$

where $\eta$ is a column matrix and $\mathbb{I}$ is an identity matrix.

**2.2** Show that

$$\Lambda_R \mathcal{B}\,(\theta,\eta)\,\Lambda_R^{-1} = \mathcal{B}\,(\theta, R\eta) \tag{10}$$

**2.3** We have argued in the class that in a relativistic quantum theory there exists a unitary operator for each Lorentz transformation that acts on states in the associated vector space. In particular, for infinitesimal Lorentz transformation (characterised by $\omega^\mu_\nu$), the operators can be expressed as

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu \qquad \text{and} \qquad U[\Lambda] = 1 - i\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu} \tag{11}$$

Find the commutation relations for the generators $M_{\mu\nu}$. We have done part of the proof in the class. We have shown that

$$\Lambda^\mu_\nu \equiv \left(\Lambda_2^{-1}\Lambda_1^{-1}\Lambda_2\Lambda_1\right)^\mu_\nu = 1 - \Big[\omega_2,\ \omega_1\Big]^\mu_\nu \tag{12}$$

By working out $U[\Lambda]$ for the above case, show that

$$\Big[M_{\mu\nu},\ M_{\rho\sigma}\Big] = i\left(g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho}\right) \tag{13}$$

**2.4** Starting with the commutation relation of generators of Lorentz transformation, show that

$$\Big[J_i, J_j\Big] = i\epsilon_{ijk}J_k\,, \tag{14}$$

where $J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$.

**2.5** Let us repeat the whole procedure for finding algebras among the generators of Poincare group, characterised by a Lorentz transformation and a translation, namely

$$x^\mu \xrightarrow[(\Lambda,a)]{} x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{15}$$

(a) Defining

$$(\Lambda_0, a_0) = (\Lambda_2, a_2)^{-1}(\Lambda_1, a_1)^{-1}(\Lambda_2, a_2)(\Lambda_1, a_1) \tag{16}$$

show that

$$(\Lambda_0)^\mu_\nu = \delta^\mu_\nu + \Big[\omega_2, \omega_1\Big]^\mu_\nu \quad \text{and} \quad a_0^\mu = (\omega_2 a_1 - \omega_1 a_2)^\mu_\nu \tag{17}$$

(b) As before, define the unitary operators for infinitesimal transformations consisting of generators (10 in total)

$$U[\Lambda, a] = 1 - i\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu} + ia^{\mu}P_{\mu} \tag{18}$$

Using the property of representations (as before, in 2.3) find the additional relations among the generators. In particular show that

$$\left[M_{\mu\nu},\ P_{\sigma}\right] = i\left(g_{\nu\sigma}P_{\mu} - g_{\mu\sigma}P_{\nu}\right) \tag{19}$$

$$\left[P_{\mu},\ P_{\nu}\right] = 0 \tag{20}$$

**2.6** Finally, denoting $P^{\mu} = \left(H, \vec{P}\right)$, show that

$$\left[J_i, H\right] = 0 \tag{21}$$

$$\left[K_i, H\right] = iP_i \tag{22}$$

$$\left[J_i, P_j\right] = i\epsilon_{ijk}P_k \tag{23}$$

$$\left[K_i, P_j\right] = i\delta_{ij}H \tag{24}$$

**2.7** Let us have some fun with spinors and sigma matrices. Using the dotted and undotted notation regarding indices of $SL\left(2, \mathcal{C}\right)$ introduced in the class show the followings:

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\,\overline{\sigma}^{\dot{\beta}\beta}_{\mu} = 2\,\delta^{\beta}_{\alpha}\,\delta^{\dot{\beta}}_{\dot{\alpha}} \tag{25}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\,\sigma_{\mu\beta\dot{\beta}} = 2\,\epsilon_{\alpha\beta}\,\epsilon_{\dot{\alpha}\dot{\beta}} \tag{26}$$

$$\left(\sigma^{\mu}\,\overline{\sigma}^{\nu} + \sigma^{\nu}\,\overline{\sigma}^{\mu}\right)_{\alpha}^{\ \beta} = 2\,g_{\mu\nu}\,\delta^{\beta}_{\alpha} \tag{27}$$

$$\sigma^{\mu}\overline{\sigma}^{\nu}\sigma^{\rho} = g^{\mu\nu}\sigma^{\rho} - g^{\mu\rho}\sigma^{\nu} + g^{\nu\rho}\sigma^{\mu} + i\epsilon^{\mu\nu\rho\kappa}\sigma_{\kappa} \tag{28}$$

$$\mathrm{Tr}\left(\sigma^{\mu}\,\overline{\sigma}^{\nu}\right) = 2\,g^{\mu\nu} \tag{29}$$

$$\mathrm{Tr}\left(\sigma^{\mu}\,\overline{\sigma}^{\nu}\,\sigma^{\rho}\,\overline{\sigma}^{\kappa}\right) = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\nu\rho}g^{\mu\kappa} + i\epsilon^{\mu\nu\rho\kappa}\right) \tag{30}$$

$$\mathrm{Tr}\left(\sigma^{\mu\nu}\right) = \mathrm{Tr}\left(\overline{\sigma}^{\mu\nu}\right) = 0 \tag{31}$$

$$\mathrm{Tr}\left(\sigma^{\mu\nu}\sigma^{\rho\kappa}\right) = \frac{1}{2}\left(g^{\mu\rho}g^{\nu\kappa} + g^{\nu\rho}g^{\mu\kappa} - i\epsilon^{\mu\nu\rho\kappa}\right) \tag{32}$$