Lecture 10 outline.

Fun with SL(2,C)

Any 4-vectors $\in \mathbb{R}^4$ (such as x^μ) can be represented as a complex 2x2 matrices.

$$X \equiv \sigma_\mu x^\mu \quad \text{or} \quad \bar{X} = \bar{\sigma}_\mu x^\mu \quad \rightarrow X, \bar{X} \text{ are Hermitian}$$

where $\sigma^\mu \equiv (\sigma_0, \vec{\sigma})$, $\bar{\sigma}^\mu \equiv (\sigma_0 - \vec{\sigma})$ $\vec{\sigma}$ are Pauli matrices

and $\sigma_{\{\mu} \bar{\sigma}_{\nu\}} = 2 g_{\mu\nu} \mathbb{1}$, $\bar{\sigma}_{\{\mu} \sigma_{\nu\}} = 2 g_{\mu\nu} \mathbb{1}$

$$\mathrm{Tr}\,[\sigma_\mu, \bar{\sigma}_\nu] = 2 g_{\mu\nu}$$

$$\Rightarrow x^2 = x_0^2 - \sum_i x_i^2 = \det X = \det \bar{X}$$

$$X\bar{X} = \bar{X}X = (\det X)\,\mathbb{1}$$

~~A transformation that preserves Hermiticity~~

$$X' = A X A^\dagger \qquad \text{if } A \in SL(2,\mathbb{C}) \Rightarrow \det A = 1$$

$\Rightarrow x'^2 = x^2$ $\longrightarrow$ leaves relativistic interval invariant

$$x^\mu \xrightarrow[\Lambda \in SO(3,1)]{} x'^\mu \quad \Rightarrow \quad X \xrightarrow[A \in SL(2,\mathbb{C})]{} X'$$

$\Rightarrow$

$$\Lambda^\mu{}_\nu = \mathrm{Tr}\,[\bar{\sigma}^\mu A \sigma_\nu A^\dagger]$$

and $A = \dfrac{e^{i\alpha}}{2\sqrt{\Lambda^\mu{}_\mu}}\, \sigma_\mu \Lambda^\mu{}_\nu \bar{\sigma}^\nu$ $\longrightarrow$ for $\det A = 1 \Rightarrow e^{i\alpha} = \pm 1$

$\Rightarrow \pm A \longrightarrow \Lambda$

Special case

(a) $A^\dagger = A^{-1} \Rightarrow \Lambda^T = \Lambda^{-1} \Rightarrow$ Rotation

(b) $A^\dagger = A \Rightarrow \Lambda^T = \Lambda \Rightarrow$ boost

Setting $\alpha = 0$, and concentrating on infinitesimal transformations $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$

$$A = \mathbb{1} - \tfrac{i}{2}\,\omega^{\mu\nu}\sigma_{\mu\nu} \qquad \sigma_{\mu\nu} = \tfrac{i}{4}\left(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu\right)$$

$$A^\dagger = \mathbb{1} + \tfrac{i}{2}\,\omega^{\mu\nu}\bar{\sigma}_{\mu\nu} \qquad \bar{\sigma}_{\mu\nu} = i/4\left(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu\right)$$

Introduce dotted index notation:

$\alpha \rightarrow$ index for A

$\dot{\alpha} \rightarrow$ index for $\bar{A} = A^*$

spinors: $\psi_\alpha \rightarrow A_\alpha{}^\beta \psi_\beta$; $\bar{\psi}^{\dot{\alpha}} \rightarrow \bar{A}^{\dot{\alpha}}{}_{\dot{\beta}}\, \bar{\psi}^{\dot{\beta}}$

$\sigma_\mu \equiv (\sigma_\mu)_{\alpha\dot{\alpha}}$; $\bar{\sigma}_\mu \equiv (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}$