Lecture 7 Outline

Lorentz transformation / Lorentz group.

for spacetime coordinates $x^\mu \equiv (x^0, x^i) \in \mathbb{R}^4$, Lorentz group is defined to be a group of transformation that leaves the relativistic interval defined via: $x^2 = g_{\mu\nu} x^\mu x^\nu$, $g_{\mu\nu} \equiv \{ g_{00} = 1,\ g_{ij} = -\delta_{ij},\ g_{0i} = g_{i0} = 0$ for $i,j = \{1,2,3\} \}$
invariant.

It follows that if $x'^\mu = x^\mu + f^\mu \Rightarrow \underbrace{\partial_\mu f_\nu + \partial_\nu f_\mu = 0}$ also $f_\mu = g_{\mu\nu} f^\nu$

$$\Downarrow$$

$$f_\mu = a_\mu + \omega_{\mu\nu} x^\nu$$

with $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$

$\Rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$

defined using $g^{\mu\sigma}$ such that $g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu$

$= \underbrace{(\delta^\mu_\nu + \omega^\mu{}_\nu)}_{\text{homogeneous Lorentz transformation.}} x^\nu + a^\mu$

From $x'^2 = x^2 \Rightarrow \boxed{\Lambda^T g \Lambda = g}$ → matrices satisfying this form a group $O(3,1)$

Covariant → $U_\mu \to U'_\mu = U_\nu (\Lambda^{-1})^\nu{}_\mu$

Contravariant → $V^\mu \to V'^\mu = \Lambda^\mu{}_\nu V^\nu$

implies $\det \Lambda = \pm$, $\Lambda^0{}_0 \geq 1$, ≤ -1 | four categories → we're interested in

$SO(3,1)^\uparrow \equiv \underbrace{\det \Lambda > 1,\ \Lambda^0{}_0 \geq 1}_{\text{continuously connected to } \mathbb{1}}$

Two important cases

(a) set of all Λ, such that $\Lambda^T = \Lambda^{-1}$

$\Rightarrow$ call $\Lambda_R \equiv \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$ $R^T R = \mathbb{1}_3$

→ form a reducible rep. of Lorentz group

(b) set of all Λ, such that $\Lambda^T = \Lambda$

call $B(\theta, \vec{\eta}) \equiv \begin{pmatrix} \cosh\theta & \sinh\theta\, \eta^T \\ \sinh\theta\, \eta & \mathbb{I}_3 + (\cosh\theta - 1)\eta\eta^T \end{pmatrix}$

→ doesn't form a group

Results: (i) Any Lorentz transformation can be written as a product of $\Lambda_R \cdot B$

(ii) Reps of Lorentz group (how it acts on Hilbert space of states) $\Rightarrow U(\Lambda) = 1 - i\omega^{\mu\nu} M_{\mu\nu}$; $\boxed{M_{\mu\nu} + M_{\nu\mu} = 0;\ M^\dagger_{\mu\nu} = M_{\mu\nu}}$

$[M_{\mu\nu}, M_{\rho\sigma}] = i\{g_{\nu\rho} M_{\mu\sigma} + g_{\mu\sigma} M_{\nu\rho} - g_{\nu\sigma} M_{\mu\rho} - g_{\mu\rho} g_{\nu\sigma}\}$